Distributed Delay-Estimation for Continuous-Time Gaussian Signals

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Information theory lab, EPFL

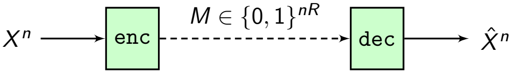
EPFL Information Processing Group

May 14, 2026
SiA group meeting

From October 2025...

Compression for the "future"

- classical compression:



- goal: to design enc, dec for small R so that $\hat{X}^n \approx X^n$, i.e., **to recover all of $\boldsymbol{X^n}$**
- often, X^n has some downstream role, e.g.,
  - prompts for language models
  - observations for distributed hypothesis testing

From October 2025... to now

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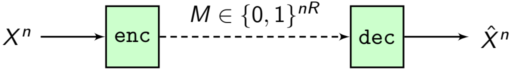

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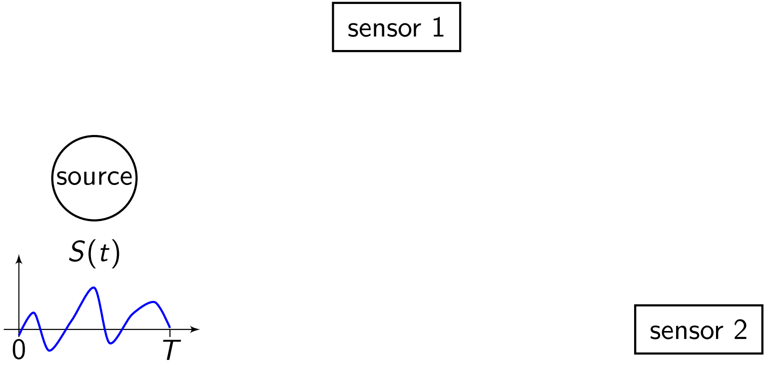
  - observations for distributed hypothesis testing
  - [NEW!] sensor observations for time-delay estimation

Outline

1. Time-delay estimation
2. Converse
3. Achievability via MIE

Outline

(Distributed) time-delay estimation



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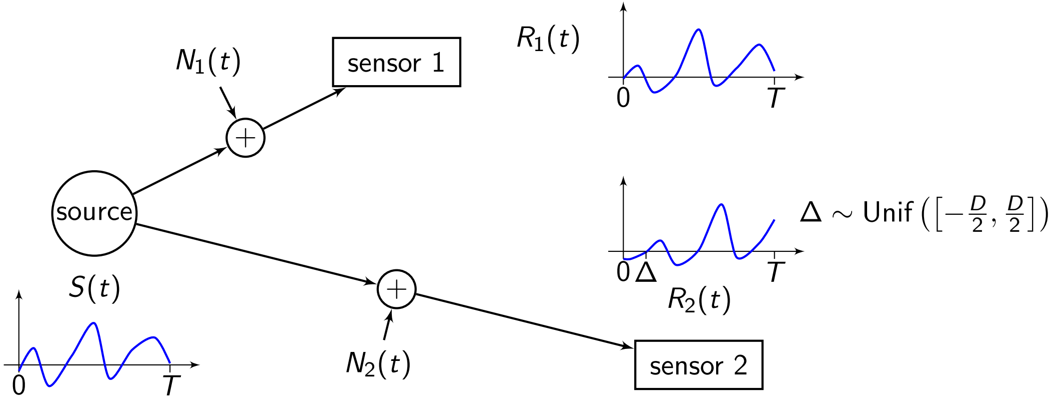

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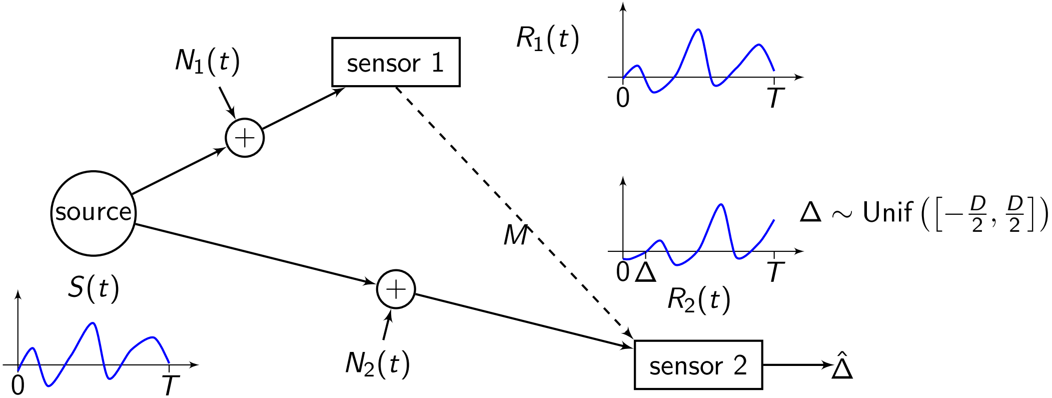

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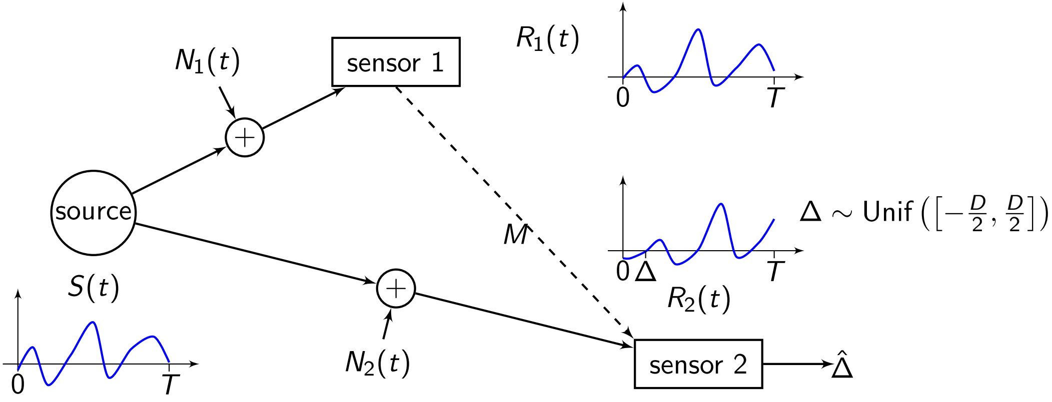

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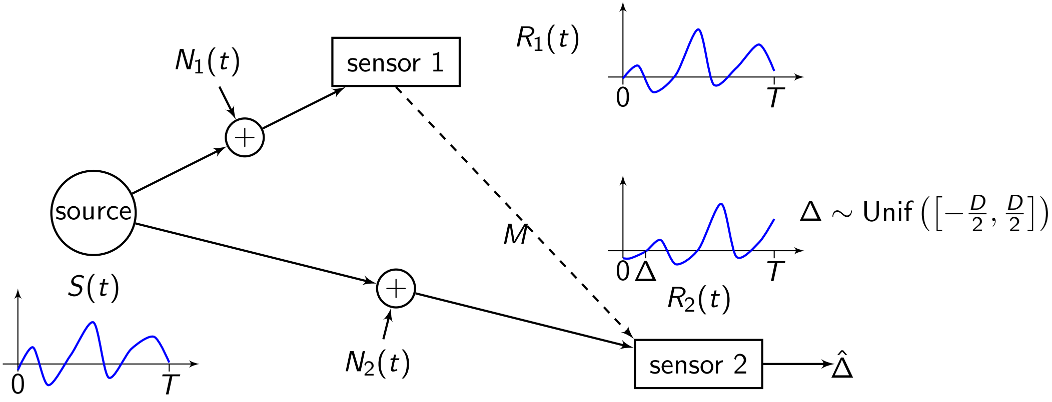

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- $\equiv \ (X, Y) : Y(t) = \rho X(t - \Delta) + \sqrt{1-\rho^2} Z(t)$ with $\rho = \frac{1}{\sqrt{(1+\sigma_1^2)(1+\sigma_2^2)}}$ and X, Z are independent, unit variance Gaussian, flat spectra on $[-\frac{W}{2}, \frac{W}{2}]$

Discrete-time version

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- $Y[n] = \rho X[n - \Delta] + \bar{\rho} Z[n]$ for $n = 1, \ldots, N$ with $\Delta \sim \text{Unif}(\{-d_m, \ldots, d_m\}) \in \mathbb{Z}$

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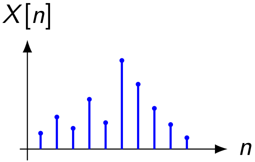

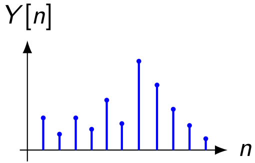

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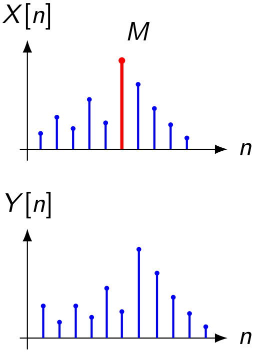

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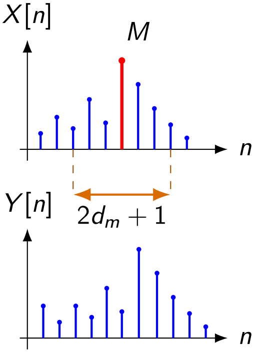

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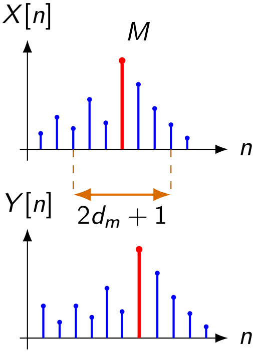

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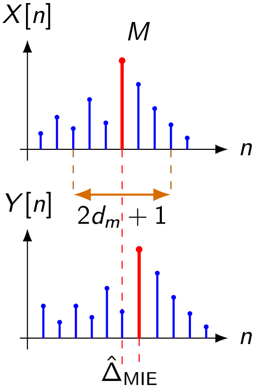

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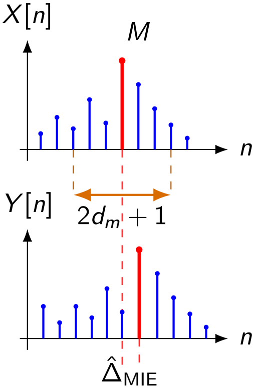

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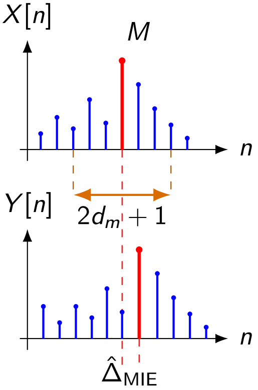

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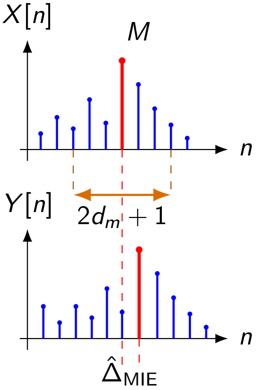

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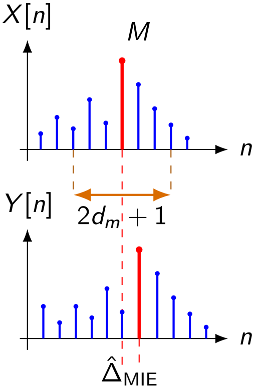

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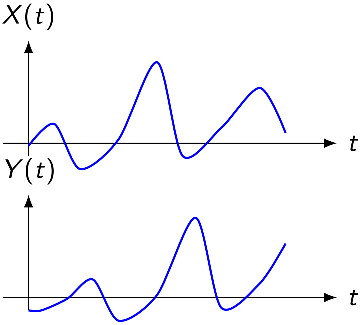

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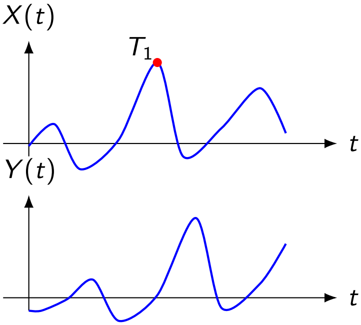

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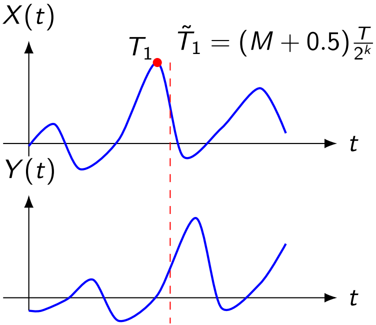

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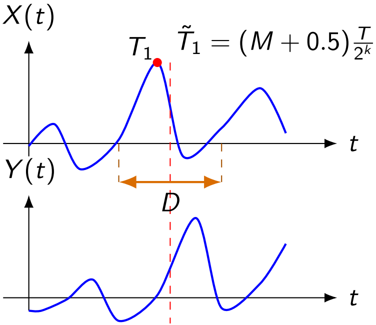

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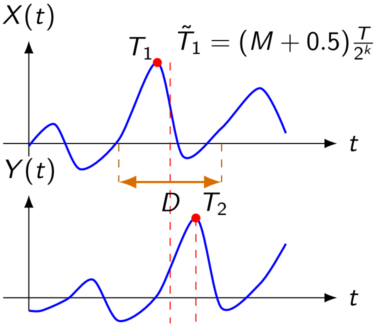

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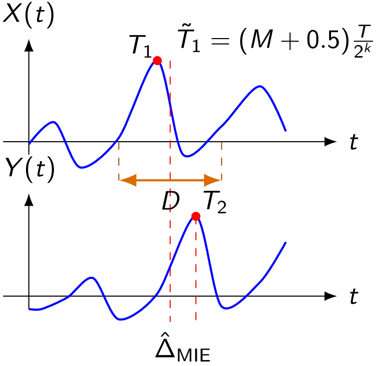

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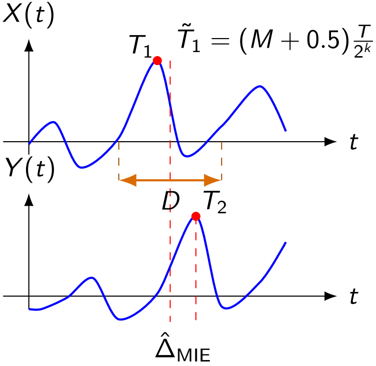

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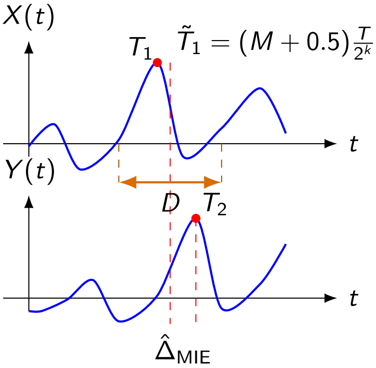

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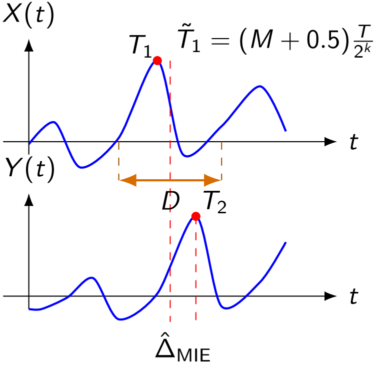

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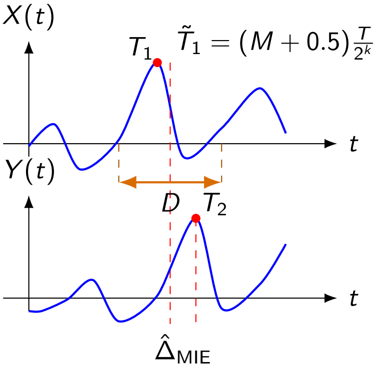

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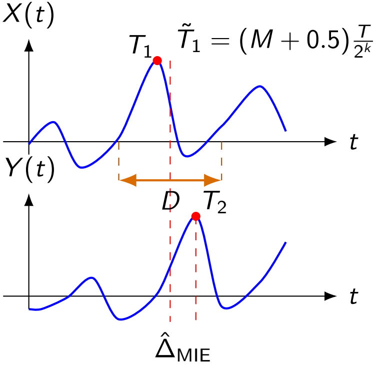

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Soln.: $\star = 1/\rho^2$ (almost)

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- Implies that for any rate $r > 1/\rho^2$, there is a constant $\gamma > 0$ such that $\Pr\{|\hat{\Delta}_{\mathsf{MIE}} - \Delta| > \gamma\} \to 0$

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Outline

Achievability proof sketch

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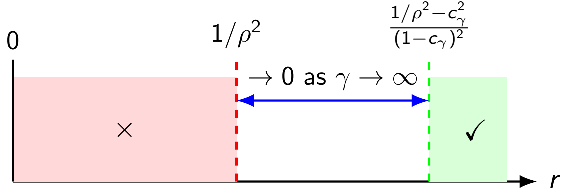

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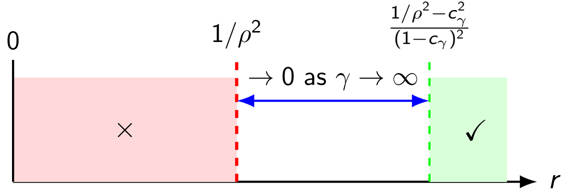

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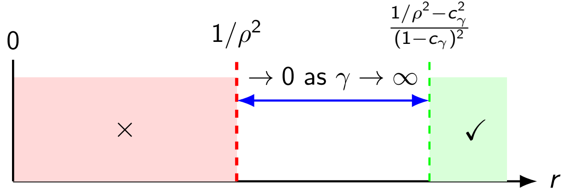

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