Entropy-Constrained Gaussian Channel Capacity

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SiA group meeting

Outline

1. Entropy-constrained Gaussian channel
2. High-SNR asymptotics
3. Low-SNR asymptotics via moment problem

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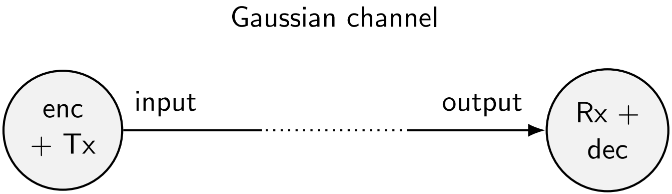
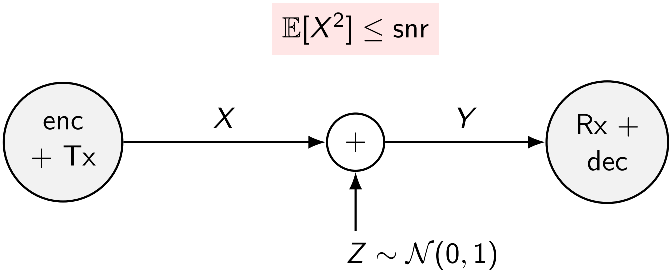
Gaussian channel



Gaussian channel

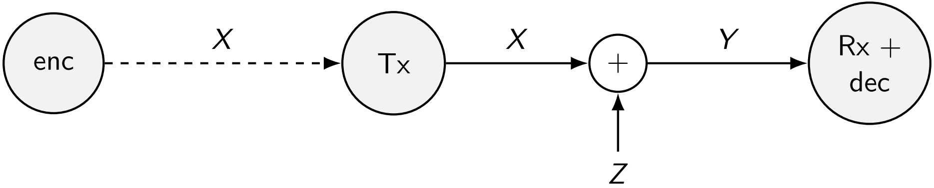


$$C(\mathsf{snr}) = \max_{X:\mathbb{E}[X^2]\leq \mathsf{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\mathsf{snr})$$

Gaussian channel

finite-capacity noiseless link

$\mathbb{E}[X^2] \leq \mathsf{snr}$



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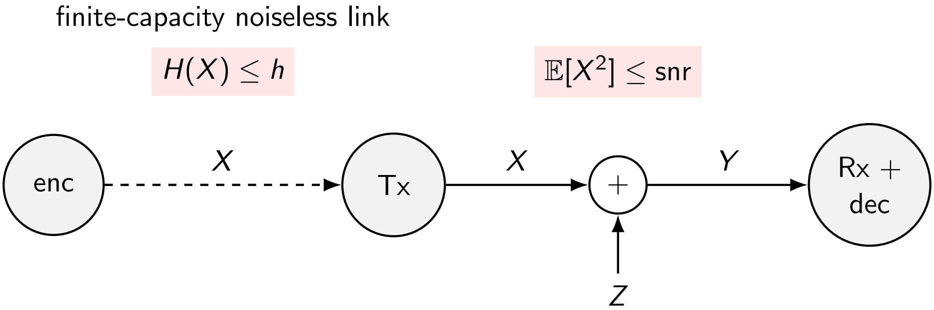
Entropy-constrained Gaussian channel



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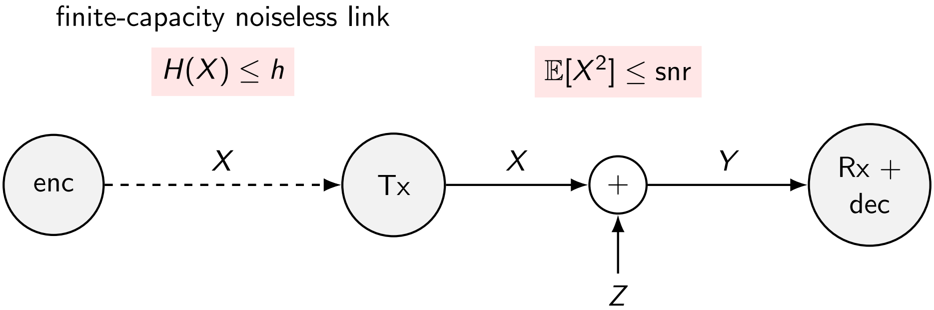
$$C_H(h,\mathsf{snr}) = \max_{X:\substack{\mathbb{E}[X^2]\leq\mathsf{snr} \\ H(X)\leq h}} I(X; X+Z)$$

Entropy-constrained Gaussian channel



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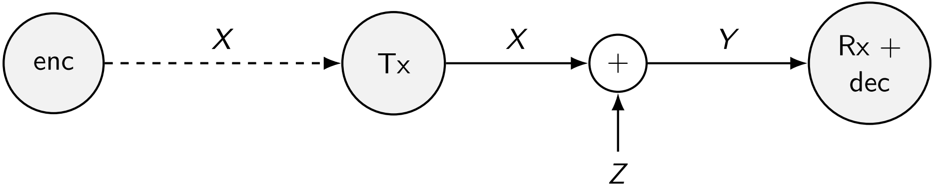
Entropy-constrained Gaussian channel

finite-capacity noiseless link

$H(X) \le h$

$\mathbb{E}[X^2] \le \mathsf{snr}$



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Approximation perspective

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- optimal X at h: discrete X with $H(X) \le h$ that is closest to $\mathcal{N}(0, 1 + \mathsf{snr})$ after "Gaussian smoothing"

Estimation perspective

Estimation perspective

- MMSE of estimating X from $Y = \sqrt{\mathsf{snr}}X + Z$:

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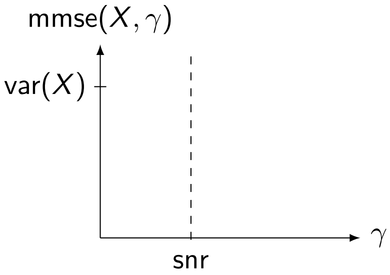

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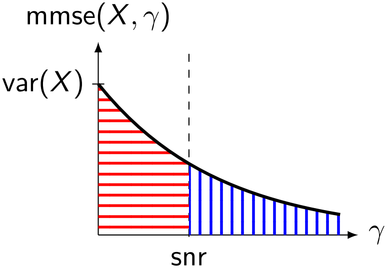

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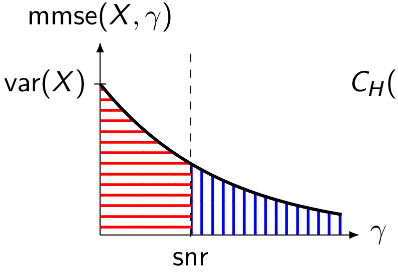

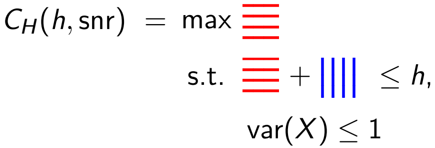

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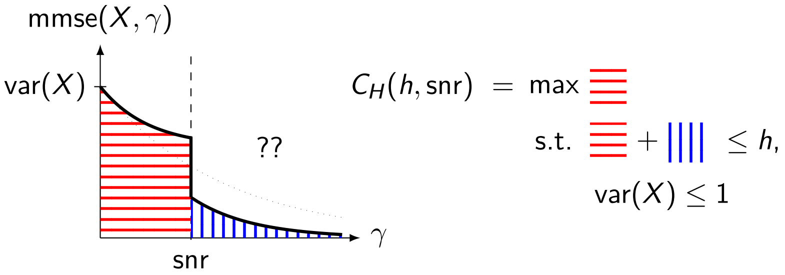

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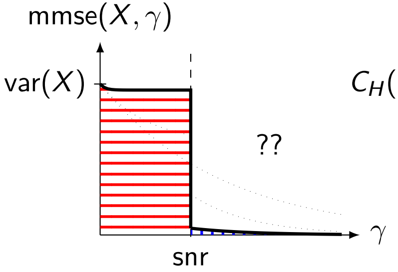

$$\begin{aligned} C_H(h, \mathsf{snr}) \;&=\; \max \\ &\quad \text{s.t.} \quad + \quad \le h, \\ &\quad \mathsf{var}(X) \le 1 \end{aligned}$$

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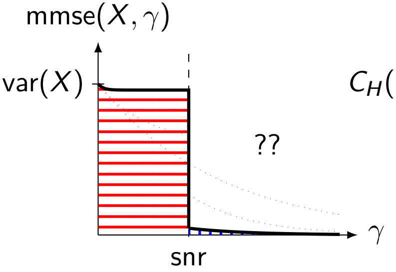

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- optimal X at snr: indistinguishable at SNR $<$ snr, distinguishable at SNR $>$ snr

C_H as a function of h and snr

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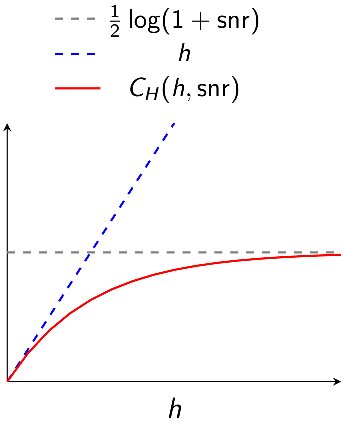

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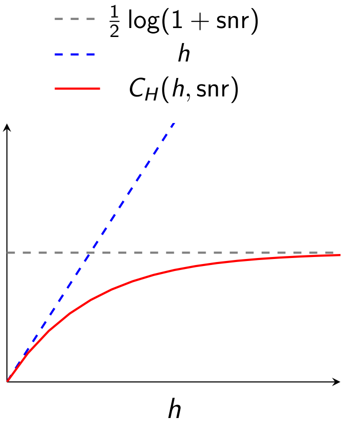

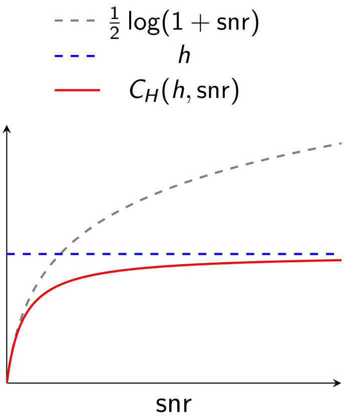

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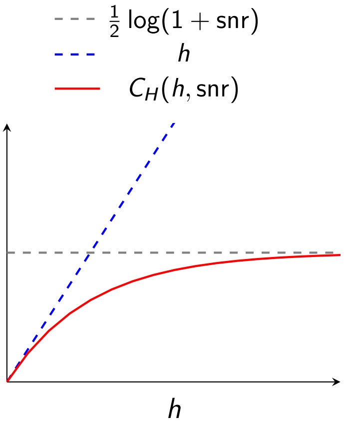

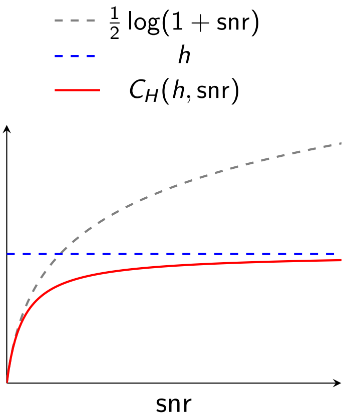

we look at asymptotic behaviour of C_H as snr $\to 0, \infty$

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High-SNR asymptotics

- $\mathsf{snr} \to \infty$: $C_H(h, \mathsf{snr}) \to h$, so gap of interest $= h - C_H(h, \mathsf{snr}) = H(X^*) - I(X^*, \mathsf{snr})$

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Lemma

For any X with $\mathbb{E}[X^2] = 1$ and finite $H(X)$, and $Z \sim \mathcal{N}(0, 1)$ independent of X, we have

$$H(X \mid \sqrt{\mathsf{snr}}X + Z) = \exp\left(-\mathsf{snr}\frac{\mathrm{d}_{\min}(X)^2}{8} + \mathrm{o}(\mathsf{snr})\right).$$

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- need stronger $X^* \xrightarrow{d} X_h$ as $\mathrm{snr} \to \infty$ to characterize **gap** in terms of $\mathrm{d}_{\min}(X_h) = \beta$

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Low SNR asymptotics via Taylor expansion

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- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$

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- $k_h :=$ maximum number of moments of G matched by X with $H(X) \leq h$
- $\boxed{C(\mathsf{snr}) - C_H(h, \mathsf{snr}) = \mathrm{O}(\mathsf{snr}^{k_h+1})}$

Aside: computing integrals numerically

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Aside: computing integrals numerically

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- given continuous $W \sim f_W$ and [C] constraints on discrete X, how large can $k_{[\mathtt{C}]}$ be?

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  finite support: at most $\lfloor k/2 \rfloor + 1$ atoms (if it exists)

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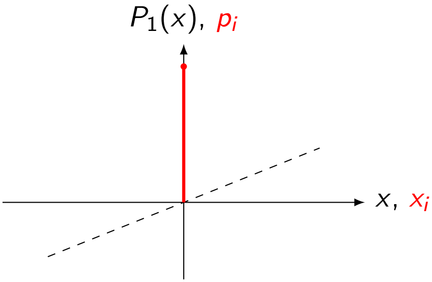

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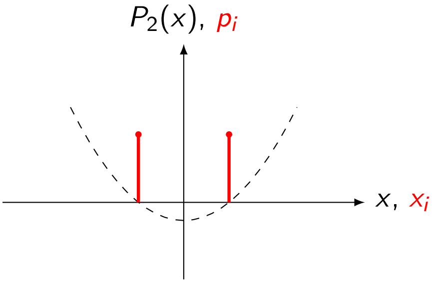

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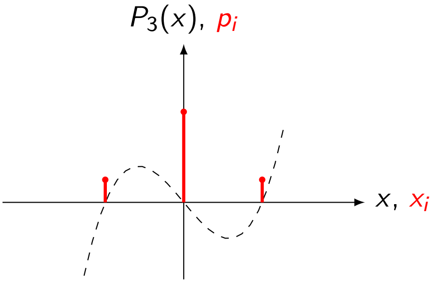

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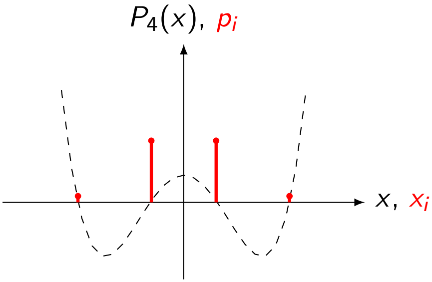

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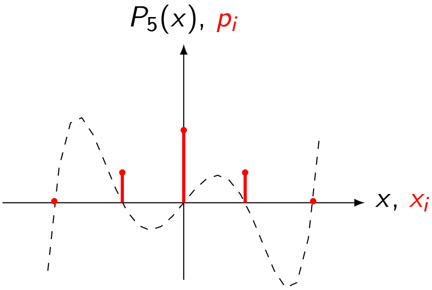

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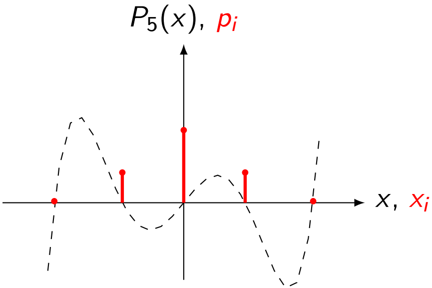

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- optimal trade-off of atoms to moments matched, but $H(X_m^{\mathrm{Q}}) \approx \frac{1}{2} \log m \to \infty$

Only three moments can be matched with small entropy

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$\eta_G = \frac{1}{3}$, *so for $h < h_2(\frac{1}{3})$, as* $\mathrm{snr} \to 0$, $C(\mathrm{snr}) - C_H(h, \mathrm{snr}) = \mathrm{O}(\mathrm{snr}^4)$.

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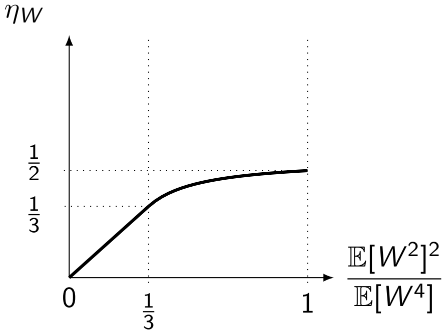

Only three moments can be matched with small entropy

Theorem

For any continuous W, there exists $\eta_W \in (0, \frac{1}{2})$ such that

(i) *if X has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is X with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

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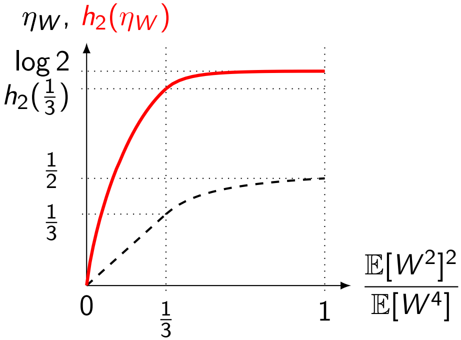

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- proof: Taylor expansion, k^{th} term depends on first k moments, can match only 3 moments

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Thank you!