

High-SNR Asymptotics of Entropy-Constrained Gaussian Channel Capacity

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joint work with Shlomo Shamai and Emre Telatar

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Entropy-constrained capacity

$$\max_X I(X; \sqrt{\text{snr}}X + Z)$$

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$$C_H(h, \text{snr}) = \max_X I(X; \sqrt{\text{snr}}X + Z)$$

over X satisfying $\mathbb{E}[X^2] \leq 1$ and $H(X) \leq h$

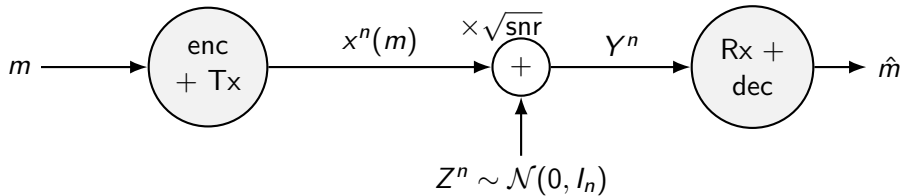
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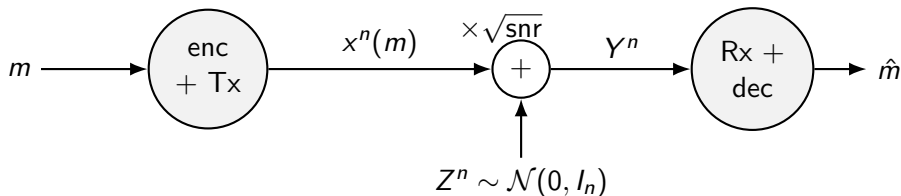
in particular: $h - C_H(h, \text{snr})$ as $\text{snr} \rightarrow \infty$

Gaussian channel



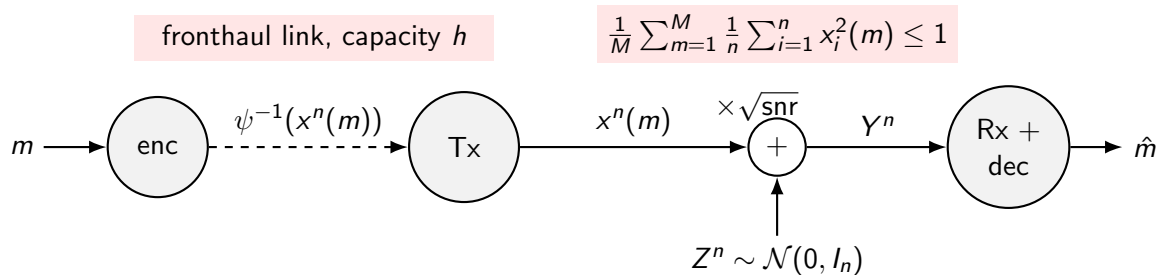
Gaussian channel

$$\frac{1}{M} \sum_{m=1}^M \frac{1}{n} \sum_{i=1}^n x_i^2(m) \leq 1$$



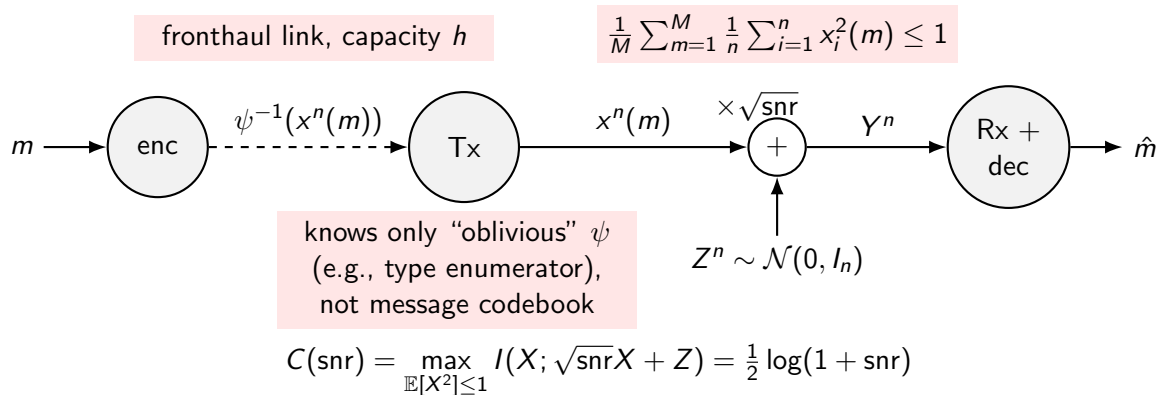
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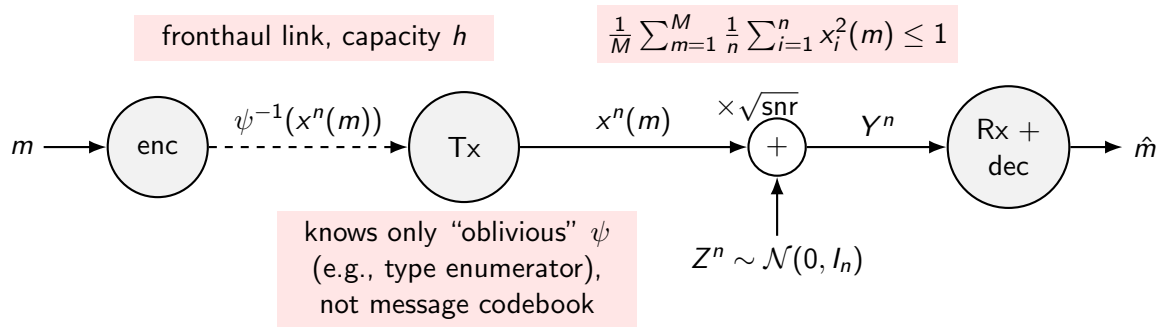


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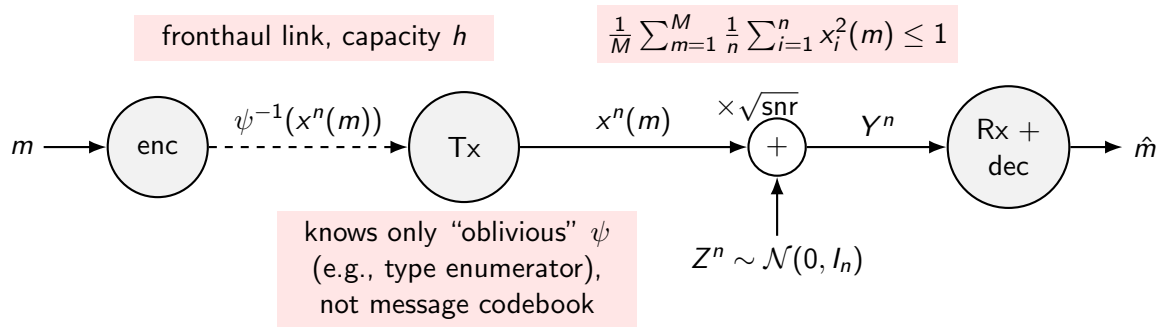
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High-SNR conditional entropy

- $\text{snr} \rightarrow \infty : C_H(h, \text{snr}) \rightarrow h$, so gap of interest $= h - C_H(h, \text{snr}) = H(X_{h,\text{snr}}^*) - I(X_{h,\text{snr}}^*, \text{snr})$

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Solution to $d_h = \max d_{\min}(X)$ over all X with $\mathbb{E}[X^2] = 1$ and $H(X) = h$ is

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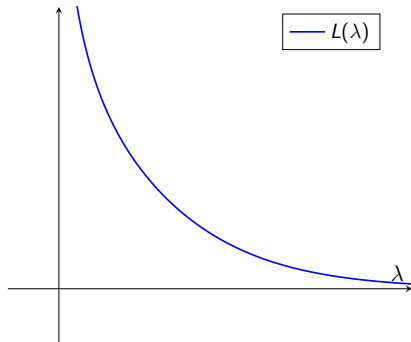
- proof sketch: maximum at equally spaced distribution, then Lagrange dual

Geometric characterization

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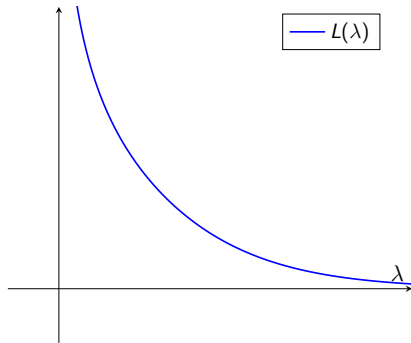
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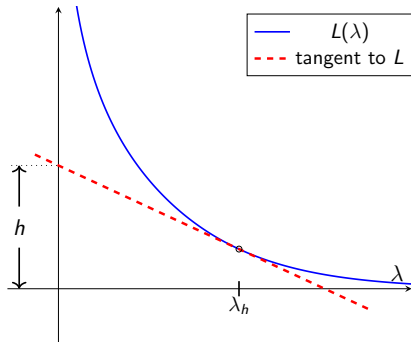
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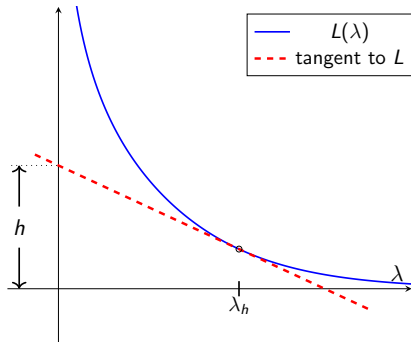
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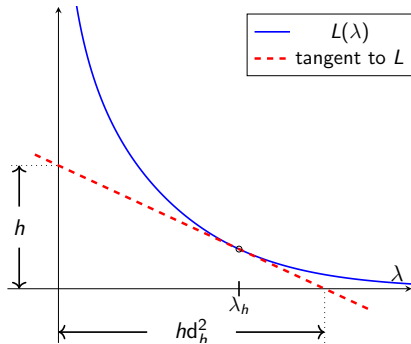
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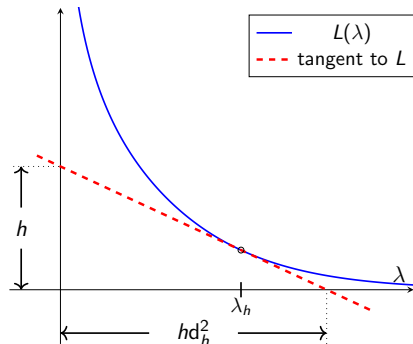
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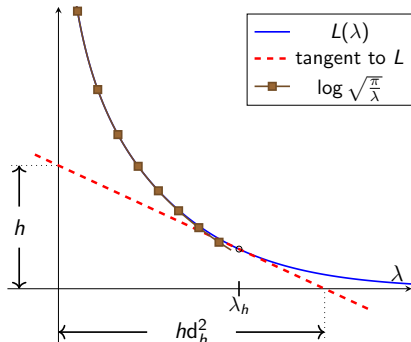
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- approximations:

Geometric characterization

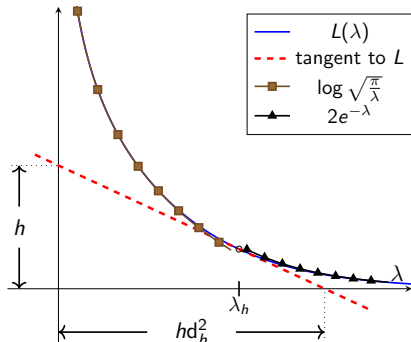
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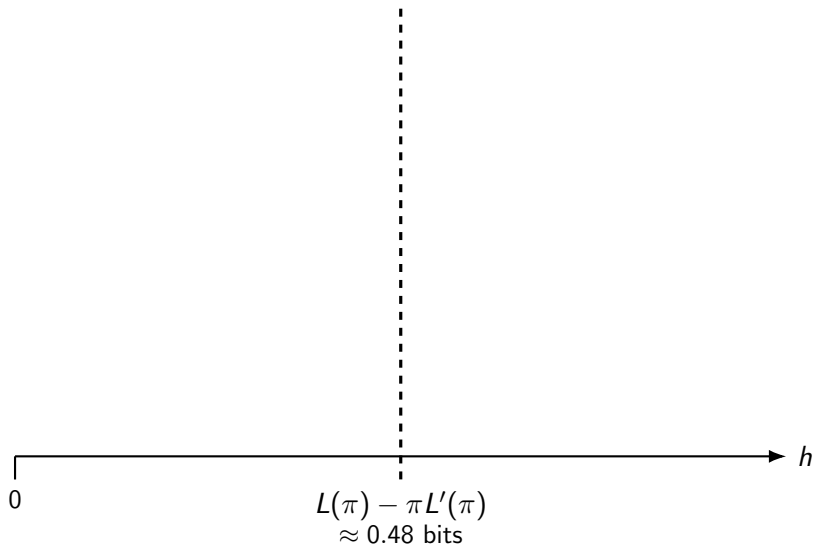


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Low- and high-entropy approximations



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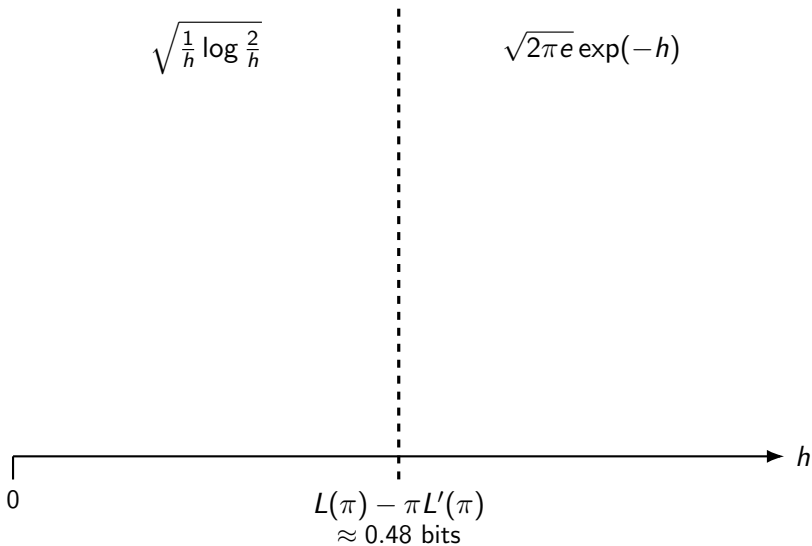


Low- and high-entropy approximations

$$d_h \approx$$

$$\sqrt{\frac{1}{h} \log \frac{2}{h}}$$

$$\sqrt{2\pi e} \exp(-h)$$



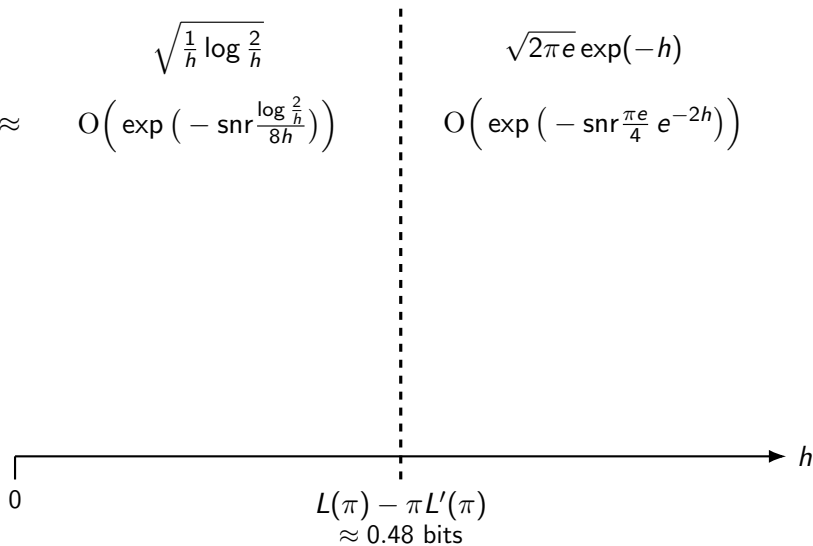
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$$d_h \approx \sqrt{\frac{1}{h} \log \frac{2}{h}}$$

$$h - C_H(h, \text{snr}) \approx O\left(\exp\left(-\text{snr} \frac{\log \frac{2}{h}}{8h}\right)\right)$$

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$$O\left(\exp\left(-\text{snr} \frac{\pi e}{4} e^{-2h}\right)\right)$$



Low- and high-entropy approximations

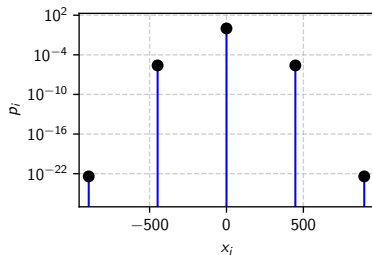
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$$L(\pi) - \pi L'(\pi) \approx 0.48 \text{ bits}$$

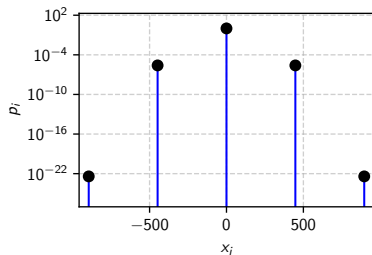
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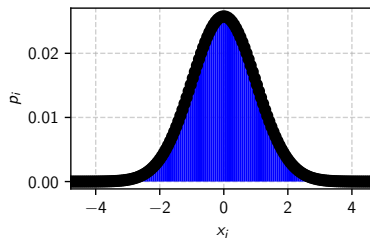
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$$\xrightarrow{h \rightarrow \infty} \mathcal{N}(0, 1)$$

0

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h

Summary

- $C_H(h, \text{snr})$, relevant in oblivious/symbol-level remote transmitters

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Thank you!