# On entropy-constrained Gaussian channel capacity via the moment problem

Adway Girish

joint work with Shlomo Shamai and Emre Telatar

EPFL

EPFL Information Processing Group

June 26, 2025

# Outline

1. Entropy-constrained Gaussian channel

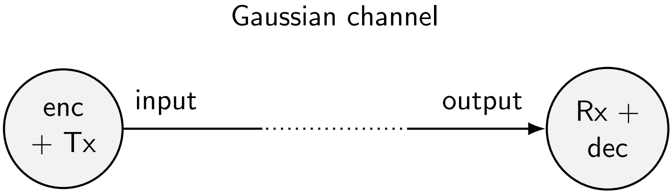
2. Moment problems

3. Low SNR capacity

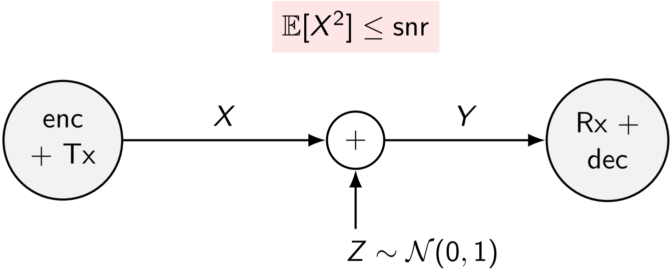
# Outline

1. Entropy-constrained Gaussian channel
2. Moment problems
3. Low SNR capacity

# Gaussian channel



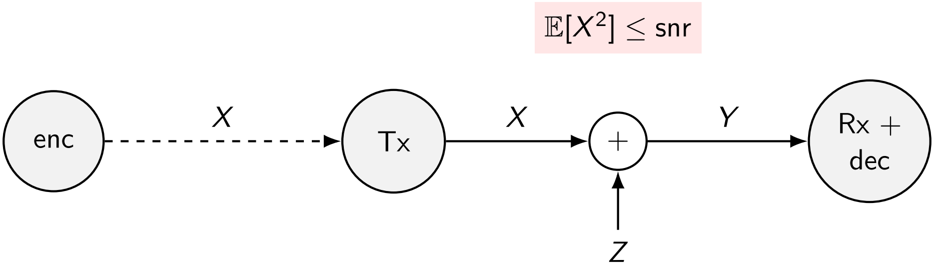
# Gaussian channel




$$C(\text{snr}) = \max_{X:\mathbb{E}[X^2]\leq \text{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\text{snr})$$

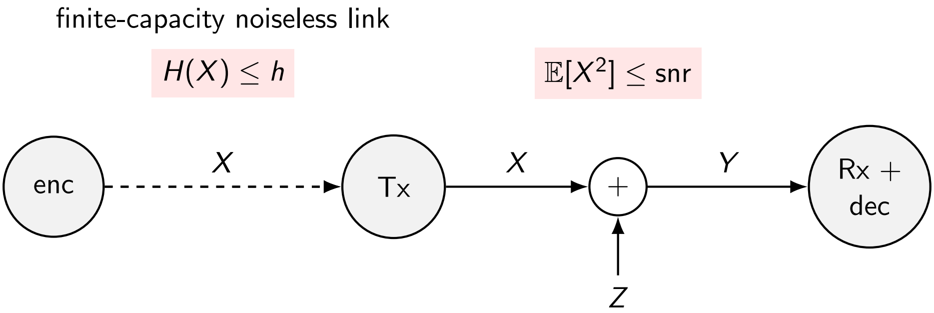
# Gaussian channel

finite-capacity noiseless link




$$C(\text{snr}) = \max_{X:\mathbb{E}[X^2]\leq \text{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\text{snr})$$

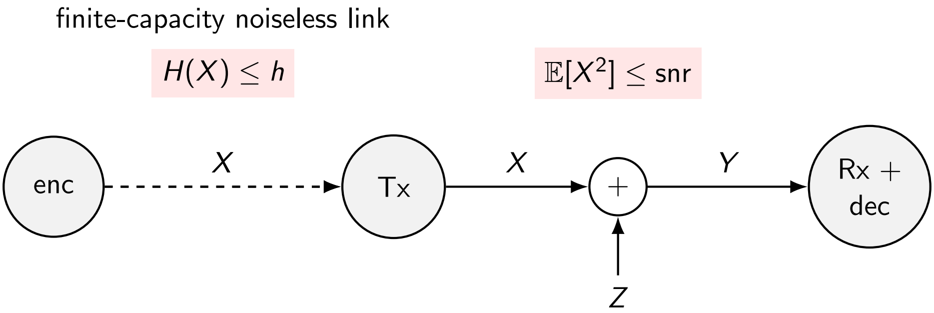
# Entropy-constrained Gaussian channel




$$C(\mathsf{snr}) = \max_{X:\mathbb{E}[X^2]\leq \mathsf{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\mathsf{snr})$$

$$C_H(h, \mathsf{snr}) = \max_{X:\substack{\mathbb{E}[X^2]\leq \mathsf{snr} \\ H(X)\leq h}} I(X; X+Z)$$

# Entropy-constrained Gaussian channel




$$C(\mathsf{snr}) = \max_{X:\mathbb{E}[X^2]\le \mathsf{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\mathsf{snr})$$

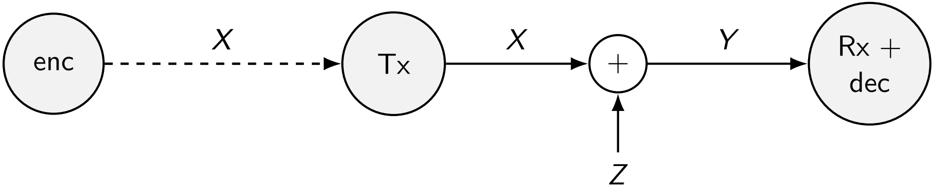
$$C_H(h,\mathsf{snr}) = \max_{X:\substack{\mathbb{E}[X^2]\le \mathsf{snr}\\ H(X)\le h}} I(X; X+Z) = \max_{X:\substack{\mathbb{E}[X^2]\le 1\\ H(X)\le h}} I(X; \sqrt{\mathsf{snr}}X+Z)$$

# Entropy-constrained Gaussian channel

finite-capacity noiseless link

$H(X) \leq h$

$\mathbb{E}[X^2] \leq \mathsf{snr}$




$$C(\mathsf{snr}) = \max_{X:\mathbb{E}[X^2]\leq \mathsf{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\mathsf{snr})$$

$$C_H(h,\mathsf{snr}) = \max_{\substack{X:\mathbb{E}[X^2]\leq \mathsf{snr} \\ H(X)\leq h}} I(X; X+Z) = \max_{\substack{X:\mathbb{E}[X^2]\leq 1 \\ H(X)\leq h}} \underbrace{I(X; \sqrt{\mathsf{snr}}X+Z)}_{I(X,\mathsf{snr})}$$

# Approximation perspective

# Approximation perspective

- wlog let $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$; $G \sim \mathcal{N}(0, 1)$ independent of $Z$

# Approximation perspective

- wlog let $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$; $G \sim \mathcal{N}(0,1)$ independent of $Z$
- $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = D(\sqrt{\mathsf{snr}}X + Z \,\|\, \sqrt{\mathsf{snr}}G + Z)$

# Approximation perspective

- wlog let $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$; $G \sim \mathcal{N}(0, 1)$ independent of $Z$
- $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = D(\sqrt{\mathsf{snr}}X + Z \,\|\, \sqrt{\mathsf{snr}}G + Z)$
- $C(\mathsf{snr}) - C_H(h, \mathsf{snr}) = \min_{X: \substack{\mathbb{E}[X^2] \le 1, \\ H(X) \le h}} D(\sqrt{\mathsf{snr}}X + Z \,\|\, \sqrt{\mathsf{snr}}G + Z)$

# Approximation perspective

- wlog let $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$; $G \sim \mathcal{N}(0,1)$ independent of $Z$
- $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = D(\sqrt{\mathsf{snr}}X + Z \,\|\, \sqrt{\mathsf{snr}}G + Z)$
- $C(\mathsf{snr}) - C_H(h, \mathsf{snr}) = \min_{X: \substack{\mathbb{E}[X^2] \leq 1, \\ H(X) \leq h}} D(\sqrt{\mathsf{snr}}X + Z \,\|\, \sqrt{\mathsf{snr}}G + Z)$
- optimal distribution at $h$: discrete $X$ with $H(X) \leq h$ that is closest to $\mathcal{N}(0, 1 + \mathsf{snr})$ after "Gaussian smoothing"

# Estimation perspective

# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:
$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

# Estimation perspective

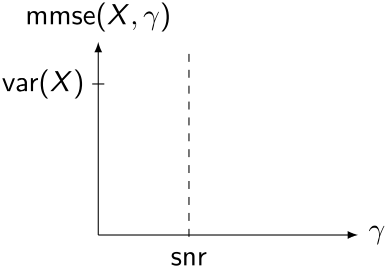
- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:
$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, \mathrm{d}\gamma, \quad H(X) = I(X, \text{``}\infty\text{''})$

# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:
$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, \mathrm{d}\gamma, \quad H(X) = I(X, \text{“}\infty\text{”})$

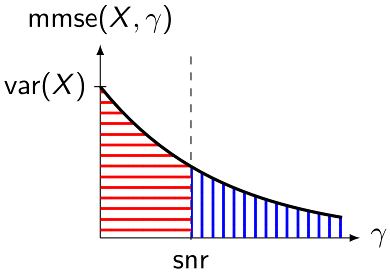


# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:

$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\,\mathrm{d}\gamma, \quad H(X) = I(X, \text{“}\infty\text{”})$

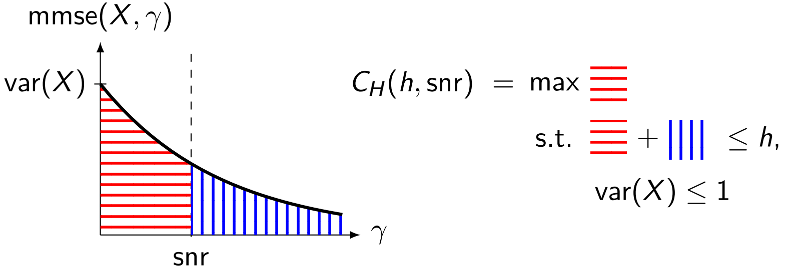


# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:

$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, \mathrm{d}\gamma, \quad H(X) = I(X, \text{“}\infty\text{”})$

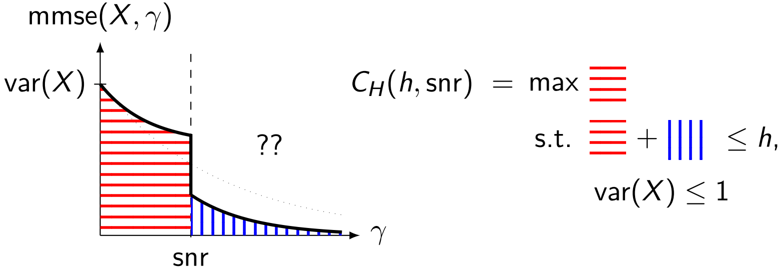


# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:

$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, \mathrm{d}\gamma, \qquad H(X) = I(X, \text{“}\infty\text{”})$

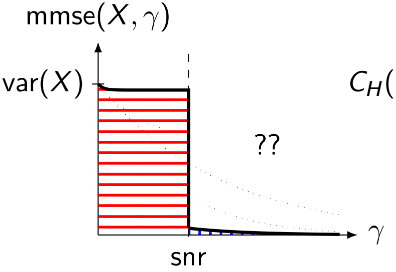


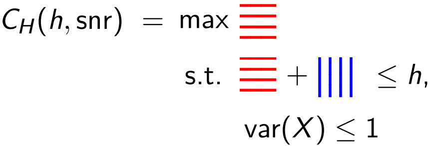
# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:

$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, d\gamma, \quad H(X) = I(X, \text{“}\infty\text{”})$



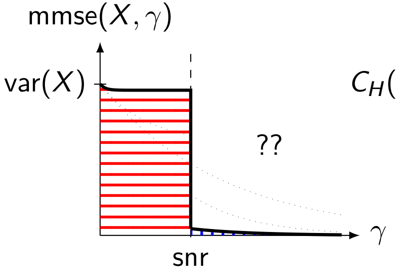



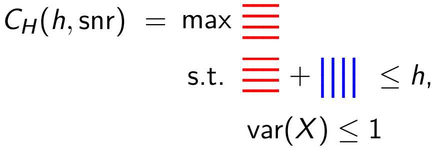
# Estimation perspective

- MMSE of estimating $X$ from $Y = \sqrt{\mathsf{snr}}X + Z$:

$$\mathsf{mmse}(X, \mathsf{snr}) = \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^2\right]$$

- I-MMSE relationship: $I(X, \mathsf{snr}) = \frac{1}{2}\int_0^{\mathsf{snr}} \mathsf{mmse}(X, \gamma)\, \mathrm{d}\gamma, \quad H(X) = I(X, \text{“}\infty\text{”})$




- optimal distribution at snr: indistinguishable at SNR $<$ snr, distinguishable at SNR $>$ snr

# Low SNR asymptotics via Taylor expansion

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X} \, \mathsf{snr}^i + r_{n,X} \, \mathsf{snr}^n$
- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \ldots, k\}$ for $k \geq 2$

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\,\mathsf{snr}^i + r_{n,X}\,\mathsf{snr}^n$
- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \ldots, k\}$ for $k \geq 2$
- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$:

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$
- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \ldots, k\}$ for $k \geq 2$
- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$: if $\mathbb{E}[X^n] = \mathbb{E}[G^n]$, $n = 1, \ldots, k$ and $\mathbb{E}[X^{k+1}] \neq \mathbb{E}[G^{k+1}]$,

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$
- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \dots, k\}$ for $k \geq 2$
- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$: if $\mathbb{E}[X^n] = \mathbb{E}[G^n]$, $n = 1, \dots, k$ and $\mathbb{E}[X^{k+1}] \neq \mathbb{E}[G^{k+1}]$, then $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = \Theta(\mathsf{snr}^{k+1})$ as $\mathsf{snr} \to 0$

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$
- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \ldots, k\}$ for $k \geq 2$
- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$: if $\mathbb{E}[X^n] = \mathbb{E}[G^n]$, $n = 1, \ldots, k$ and $\mathbb{E}[X^{k+1}] \neq \mathbb{E}[G^{k+1}]$, then $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = \Theta(\mathsf{snr}^{k+1})$ as $\mathsf{snr} \to 0$
- $k_h :=$ maximum number of moments of $G$ matched by $X$ with $H(X) \leq h$

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$

- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \ldots, k\}$ for $k \geq 2$

- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$: if $\mathbb{E}[X^n] = \mathbb{E}[G^n]$, $n = 1, \ldots, k$ and $\mathbb{E}[X^{k+1}] \neq \mathbb{E}[G^{k+1}]$, then $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = \Theta(\mathsf{snr}^{k+1})$ as $\mathsf{snr} \to 0$

- $k_h :=$ maximum number of moments of $G$ matched by $X$ with $H(X) \leq h$

- $\boxed{C(\mathsf{snr}) - C_H(h, \mathsf{snr}) = \mathcal{O}(\mathsf{snr}^{k_h+1})}$

# Low SNR asymptotics via Taylor expansion

- $\mathbb{E}[X^{2n}] < \infty$: $I(X, \mathsf{snr}) = \sum_{i=1}^{n-1} a_{i,X}\, \mathsf{snr}^i + r_{n,X}\, \mathsf{snr}^n$

- $a_{k,X}$ is a polynomial of $\{\mathbb{E}[X^n] : n = 1, \dots, k\}$ for $k \geq 2$

- for $X$ with $\mathbb{E}[X^{2(k+2)}] < \infty$: if $\mathbb{E}[X^n] = \mathbb{E}[G^n]$, $n = 1, \dots, k$ and $\mathbb{E}[X^{k+1}] \neq \mathbb{E}[G^{k+1}]$, then $I(G, \mathsf{snr}) - I(X, \mathsf{snr}) = \Theta(\mathsf{snr}^{k+1})$ as $\mathsf{snr} \to 0$

- $k_h :=$ maximum number of moments of $G$ matched by $X$ with $H(X) \leq h$

- $\boxed{C(\mathsf{snr}) - C_H(h, \mathsf{snr}) = \mathcal{O}(\mathsf{snr}^{k_h+1})}$
  ($\mathcal{O}$ instead of $\Theta$ to allow for $X$ with $\mathbb{E}[X^{2(k_h+2)}] = \infty$)

# Outline


1 Entropy-constrained Gaussian channel

2 Moment problems

3 Low SNR capacity

# Classical moment problem

# Classical moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

# Classical moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0$ for $n = 1, 2, 3, \ldots$

# Classical moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0$ for $n = 1, 2, 3, \ldots$;

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

# Truncated moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $$\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0 \text{ for } n = 1, 2, 3, \ldots;$$

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

- Q: Given $s_1, s_2, s_3, \ldots, s_k$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots, k$?

# Truncated moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $$\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0 \text{ for } n = 1, 2, 3, \ldots;$$

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

- Q: Given $s_1, s_2, s_3, \ldots, s_k$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots, k$?

  A: iff ($k$ odd)

  ($k$ even)

# Truncated moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0$ for $n = 1, 2, 3, \ldots$;

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

- Q: Given $s_1, s_2, s_3, \ldots, s_k$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots, k$?

  A: iff ($k$ odd) there exists $\tilde{s}_{k+1}$ such that $\mathrm{H}_{\frac{k+1}{2}}(s_1, \ldots, s_k, \tilde{s}_{k+1}) \succeq 0$

  ($k$ even)

# Truncated moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $$\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0 \text{ for } n = 1, 2, 3, \ldots;$$

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

- Q: Given $s_1, s_2, s_3, \ldots, s_k$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots, k$?

  A: iff ($k$ odd) there exists $\tilde{s}_{k+1}$ such that $\mathrm{H}_{\frac{k+1}{2}}(s_1, \ldots, s_k, \tilde{s}_{k+1}) \succeq 0$

  ($k$ even) there exist $\tilde{s}_{k+1}, \tilde{s}_{k+2}$ such that $\mathrm{H}_{\frac{k}{2}+1}(s_1, \ldots, s_k, \tilde{s}_{k+1}, \tilde{s}_{k+2}) \succeq 0$

# Truncated moment problem

- Q: Given $s_1, s_2, s_3, \ldots$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots$?

  A: iff $$\mathrm{H}_n(s_1, \ldots, s_{2n}) = \begin{pmatrix} 1 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix} \succeq 0 \text{ for } n = 1, 2, 3, \ldots;$$

  infinite support iff $\mathrm{H}_n \succ 0$ for all $n$

- Q: Given $s_1, s_2, s_3, \ldots, s_k$, does there exist $X$ on $\mathbb{R}$ such that $\mathbb{E}[X^n] = s_n$ for $n = 1, 2, 3, \ldots, k$?

  A: iff ($k$ odd) there exists $\tilde{s}_{k+1}$ such that $\mathrm{H}_{\frac{k+1}{2}}(s_1, \ldots, s_k, \tilde{s}_{k+1}) \succeq 0$

  ($k$ even) there exist $\tilde{s}_{k+1}, \tilde{s}_{k+2}$ such that $\mathrm{H}_{\frac{k}{2}+1}(s_1, \ldots, s_k, \tilde{s}_{k+1}, \tilde{s}_{k+2}) \succeq 0$;

  finite support: at most $\lfloor k/2 \rfloor + 1$ atoms (if it exists)

# Outline

1 Entropy-constrained Gaussian channel

2 Moment problems

3 Low SNR capacity

# Main result

**Theorem**

*For any continuous* $W$*, there exists* $\eta_W \in (0, \frac{1}{2})$ *such that*

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

# Main result

**Theorem**

*For any continuous* $W$*, there exists* $\eta_W \in (0, \frac{1}{2})$ *such that*

(i) *if* $X$ *has* $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ *for* $n = 1, 2, 3, 4$*, then* $H(X) \geq h_2(\eta_W)$*.*

(ii) *for any* $h > 0$*, there is* $X$ *with* $H(X) \leq h$ *and* $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ *for* $n = 1, 2, 3$*.*

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

## Proof idea

□

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

## Proof idea

- $H(X) \leq h < \log 2 \iff X = \begin{cases} \tilde{X} & \text{w.p. } \epsilon < 1/2 \\ x_0 & \text{w.p. } 1 - \epsilon > 1/2 \end{cases}$

□

# Main result

**Theorem**

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

**Proof idea**

- $H(X) \leq h < \log 2 \iff X = \begin{cases} \tilde{X} & \text{w.p. } \epsilon < 1/2 \\ x_0 & \text{w.p. } 1 - \epsilon > 1/2 \end{cases}$
- need $\mathbb{E}[\tilde{X}^n] = s_n := \frac{1}{\epsilon}(\mathbb{E}[W^n] - (1-\epsilon)x_0^n)$

□

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

## Proof idea

- $H(X) \leq h < \log 2 \iff X = \begin{cases} \tilde{X} & \text{w.p. } \epsilon < 1/2 \\ x_0 & \text{w.p. } 1 - \epsilon > 1/2 \end{cases}$
- need $\mathbb{E}[\tilde{X}^n] = s_n := \frac{1}{\epsilon}(\mathbb{E}[W^n] - (1-\epsilon)x_0^n)$
- check: $s_1, \ldots, s_4$ "valid" iff $\epsilon > \eta_W$, but $s_1, s_2, s_3$ always "valid" □

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

## Proof idea

- $H(X) \leq h < \log 2 \iff X = \begin{cases} \tilde{X} & \text{w.p. } \epsilon < 1/2 \\ x_0 & \text{w.p. } 1 - \epsilon > 1/2 \end{cases}$
- need $\mathbb{E}[\tilde{X}^n] = s_n := \frac{1}{\epsilon}(\mathbb{E}[W^n] - (1-\epsilon)x_0^n)$
- check: $s_1, \ldots, s_4$ "valid" iff $\epsilon > \eta_W$, but $s_1, s_2, s_3$ always "valid" □

## Corollary

$\eta_G = \frac{1}{3}$

# Main result

## Theorem

*For any continuous $W$, there exists $\eta_W \in (0, \frac{1}{2})$ such that*

(i) *if $X$ has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is $X$ with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

## Proof idea

- $H(X) \leq h < \log 2 \iff X = \begin{cases} \tilde{X} & \text{w.p. } \epsilon < 1/2 \\ x_0 & \text{w.p. } 1 - \epsilon > 1/2 \end{cases}$
- need $\mathbb{E}[\tilde{X}^n] = s_n := \frac{1}{\epsilon}(\mathbb{E}[W^n] - (1-\epsilon)x_0^n)$
- check: $s_1, \ldots, s_4$ "valid" iff $\epsilon > \eta_W$, but $s_1, s_2, s_3$ always "valid" □

## Corollary

*$\eta_G = \frac{1}{3}$, so for $h < h_2(\frac{1}{3})$, as* $\text{snr} \to 0$, $C(\text{snr}) - C_H(h, \text{snr}) = \mathcal{O}(\text{snr}^4)$.

# Summary

- entropy-constrained Gaussian channel

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathsf{snr})$

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathsf{snr})$
- $\mathsf{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathsf{snr}) = C(\mathsf{snr}) - \mathcal{O}(\mathsf{snr}^4)$

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \text{snr})$
- $\text{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \text{snr}) = C(\text{snr}) - \mathcal{O}(\text{snr}^4)$, via entropy-constrained version of truncated moment problem

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathrm{snr})$
- $\mathrm{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathrm{snr}) = C(\mathrm{snr}) - \mathcal{O}(\mathrm{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathsf{snr})$
- $\mathsf{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathsf{snr}) = C(\mathsf{snr}) - \mathcal{O}(\mathsf{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy
- open:

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathrm{snr})$
- $\mathrm{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathrm{snr}) = C(\mathrm{snr}) - \mathcal{O}(\mathrm{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy
- open:
    - structure of capacity-achieving distributions

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathrm{snr})$
- $\mathrm{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathrm{snr}) = C(\mathrm{snr}) - \mathcal{O}(\mathrm{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy
- open:
  - structure of capacity-achieving distributions
  - $C_H(h, \mathrm{snr})$ for other $h, \mathrm{snr}$

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathsf{snr})$
- $\mathsf{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathsf{snr}) = C(\mathsf{snr}) - \mathcal{O}(\mathsf{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy
- open:
    - structure of capacity-achieving distributions
    - $C_H(h, \mathsf{snr})$ for other $h, \mathsf{snr}$
    - insights from/to estimation problems?

# Summary

- entropy-constrained Gaussian channel capacity $C_H(h, \mathrm{snr})$
- $\mathrm{snr} \to 0$, $0 < h < h_2(\frac{1}{3})$: $C_H(h, \mathrm{snr}) = C(\mathrm{snr}) - \mathcal{O}(\mathrm{snr}^4)$, via entropy-constrained version of truncated moment problem
- for any continuous distribution, only three moments can be matched by a discrete distribution of sufficiently small entropy
- open:
  - structure of capacity-achieving distributions
  - $C_H(h, \mathrm{snr})$ for other $h, \mathrm{snr}$
  - insights from/to estimation problems?

Thank you!