Compression and Contraction

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EPFL Information Processing Group

September 9, 2024
IPG PhD Review

Outline

1. Prompt compression for black-box language models
2. Input-entropy-constrained capacity
3. Joint range of divergences
4. Closing remarks

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2. Input-entropy-constrained capacity
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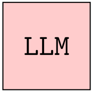
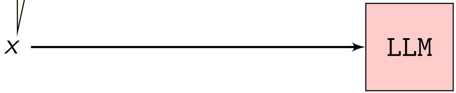
Prompt compression



Prompt compression

Prompt

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of light, it was the season of darkness, it was the spring of hope, it was the winter of despair.



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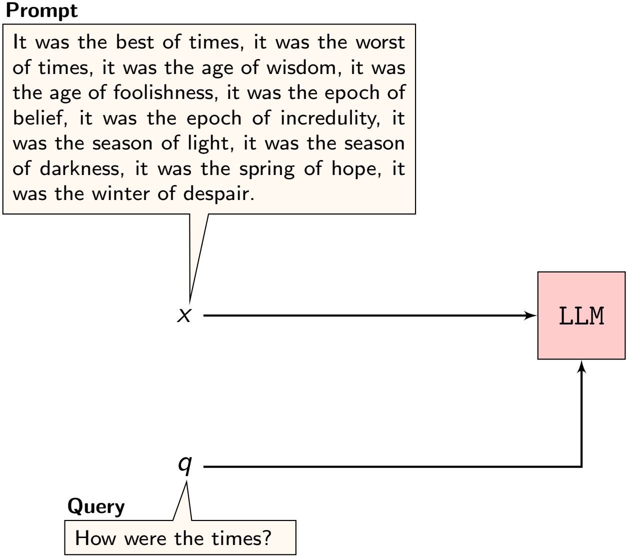

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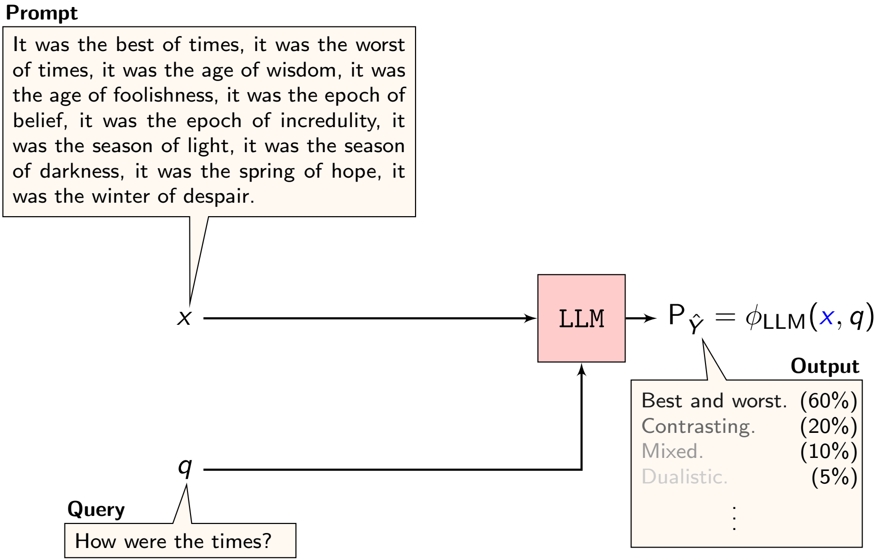
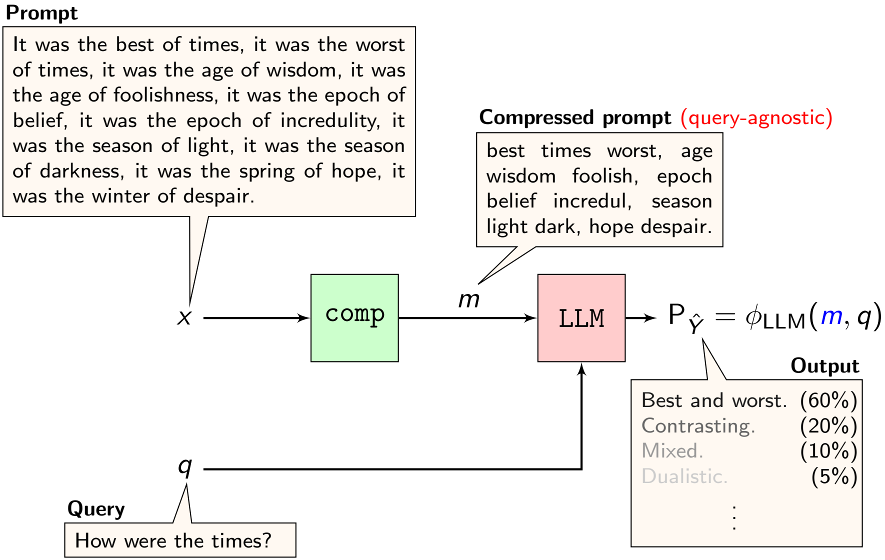
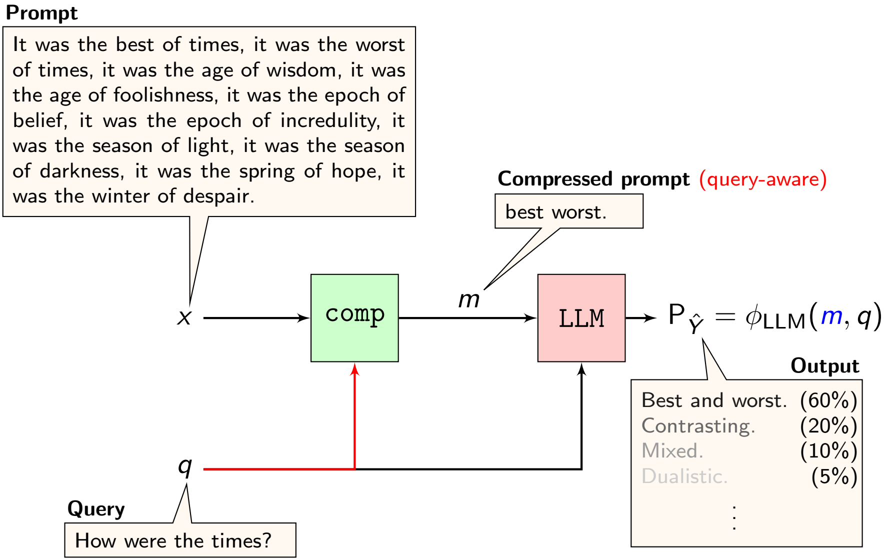


Prompt compression: query-agnostic



Prompt compression: query-aware



Prompt compression: rate-distortion formulation

- $(X, Q, Y) \sim \mathrm{P}_{XQY} = \mathrm{P}_{XQ}\,\mathrm{P}_{Y|XQ}$ $\qquad Y =$ “true answer”

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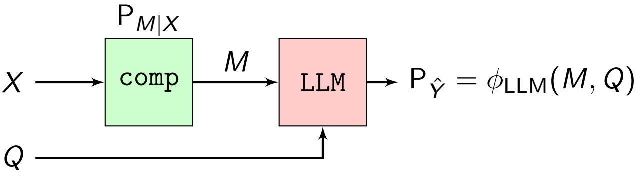

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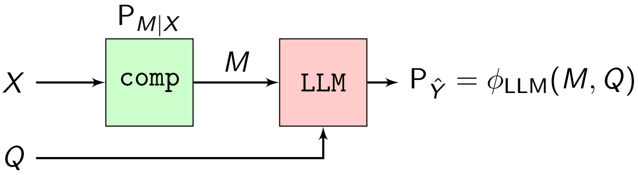

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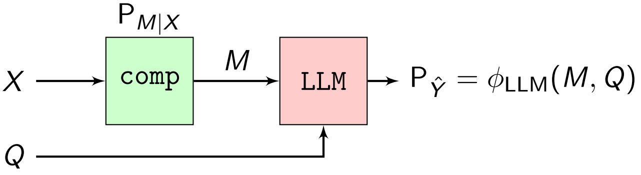

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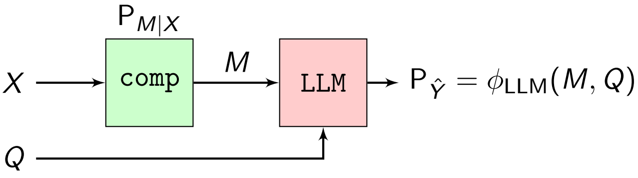

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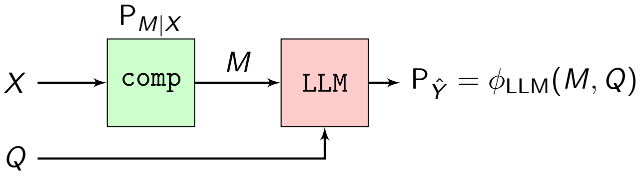

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$$\text{rate} = \mathbb{E}\left[\frac{\mathrm{len}(M)}{\mathrm{len}(X)}\right] \qquad \text{distortion} = \mathbb{E}\left[\mathrm{d}\big(Y, \phi_{\mathsf{LLM}}(M, Q)\big)\right]$$

Distortion-rate function

- $$\text{rate} = \mathbb{E}\left[\frac{\text{len}(M)}{\text{len}(X)}\right] \qquad \text{distortion} = \mathbb{E}\left[\text{d}(Y, \phi_{\text{LLM}}(M, Q))\right]$$

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"normalized" distortion, rate on compressing $x \mapsto m$

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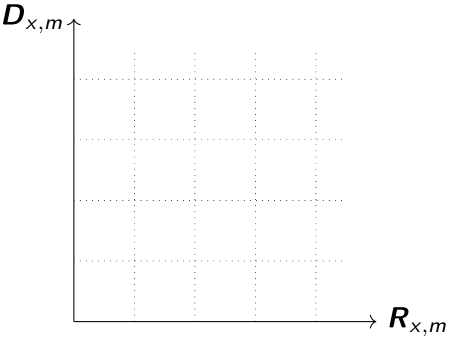

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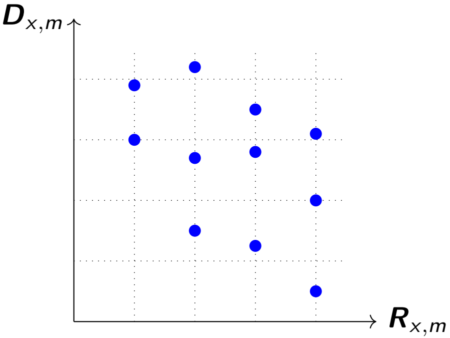

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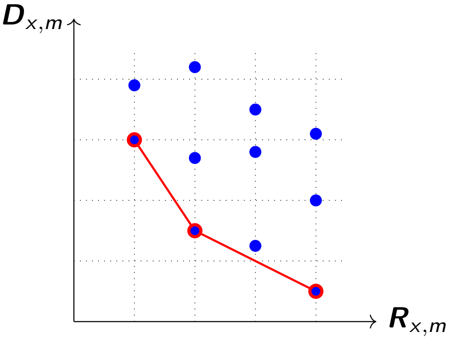

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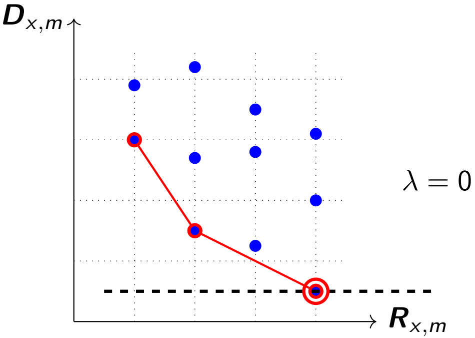

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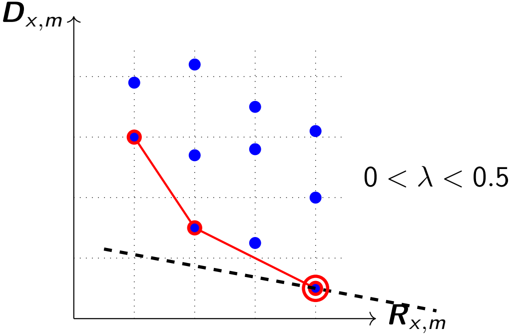

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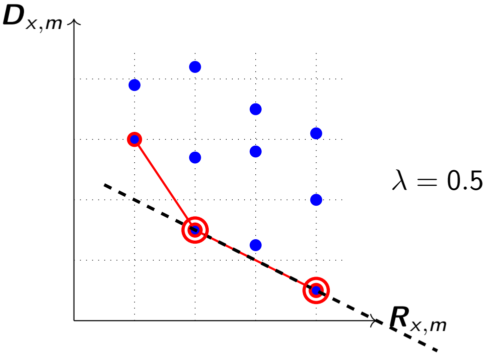

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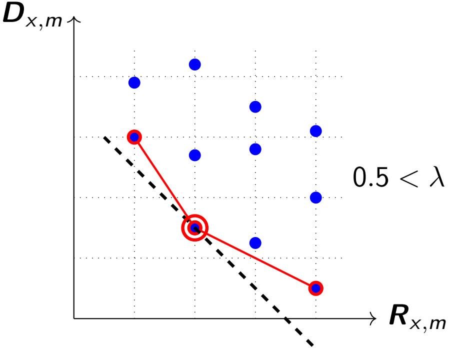

$0.5 < \lambda < 1.5$

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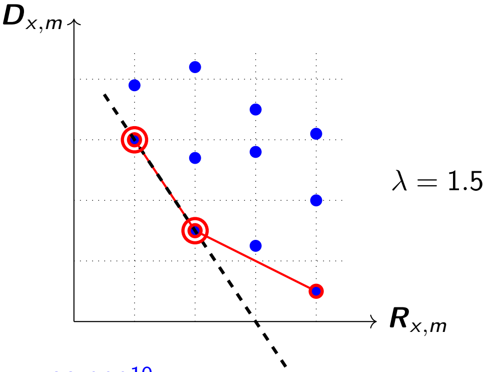

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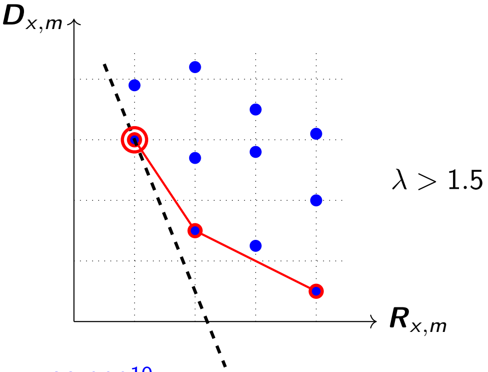

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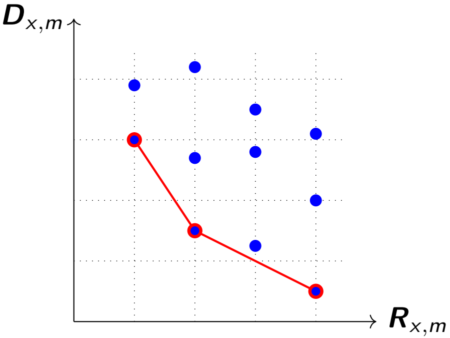

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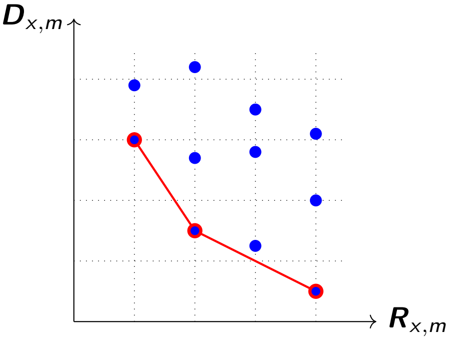

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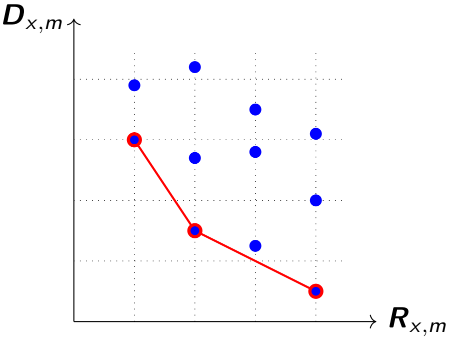

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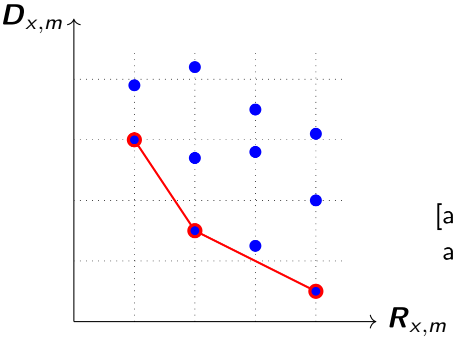

[apple $\mapsto$ app, ale, pe;
apple $\not\mapsto$ pale, red, lp]

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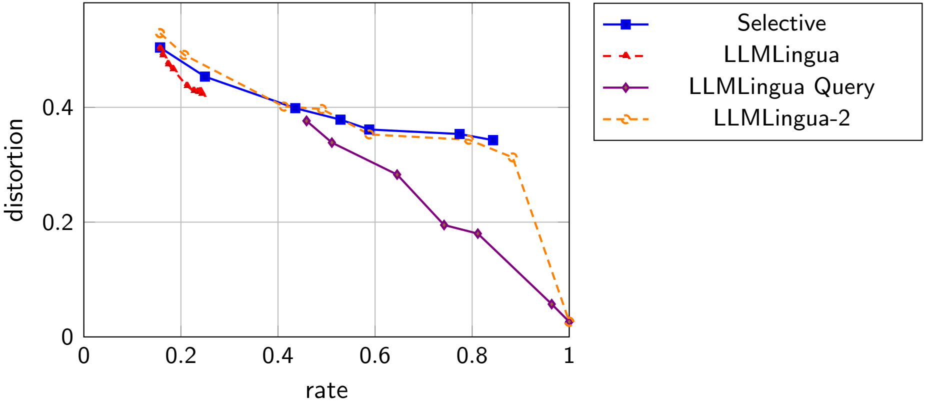
Experimental results



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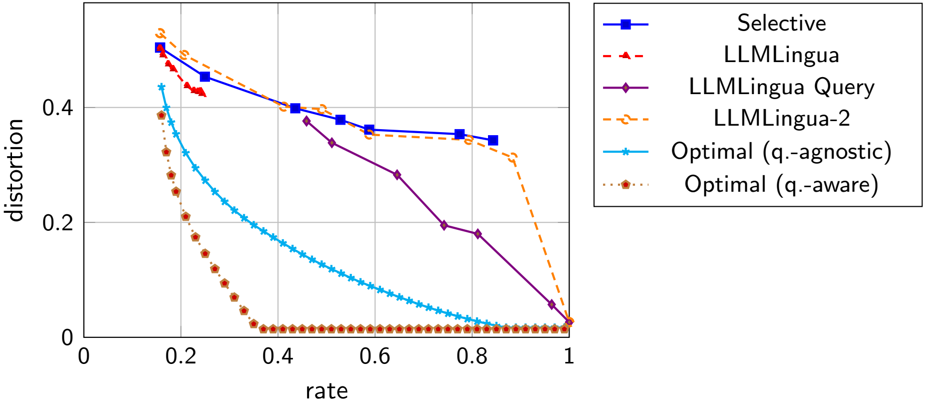

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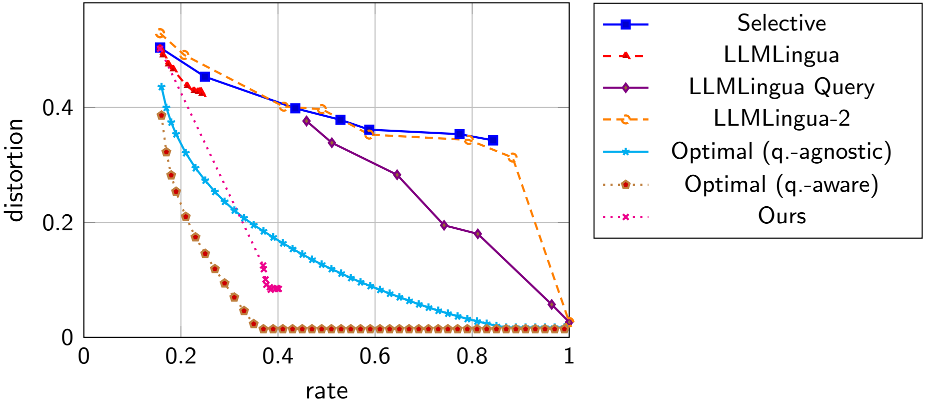

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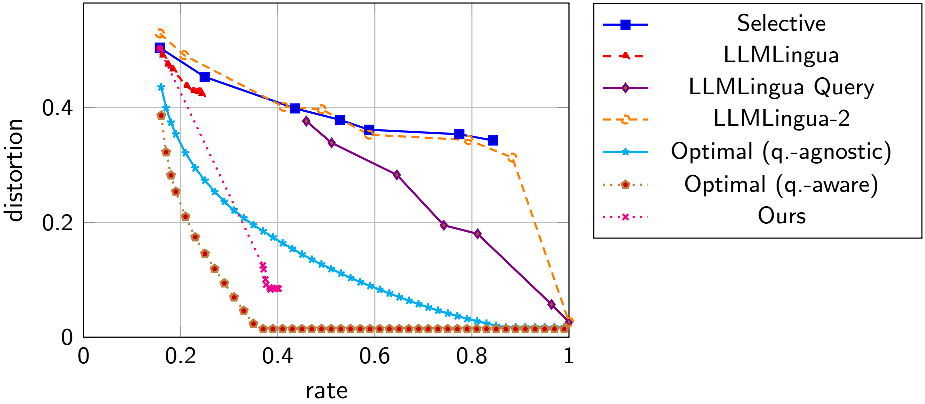

A.G.*, A.Nagle*, M.Bondaschi, M.Gastpar, A.V.Makkuva, H.Kim, "Fundamental Limits of Prompt Compression: A Rate-Distortion Framework for Black-Box Language Models."
— ICML 2024 Workshop on Theoretical Foundations of Foundation Models **[Oral]**
— under review at `[conference]`

Segue to a contraction problem

- Optimization 101. . .

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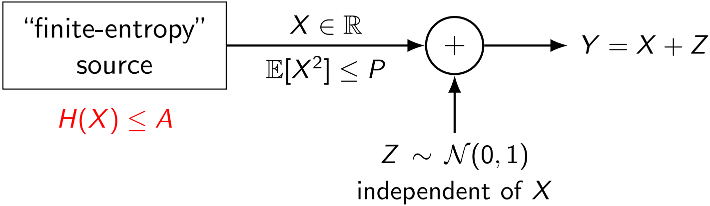
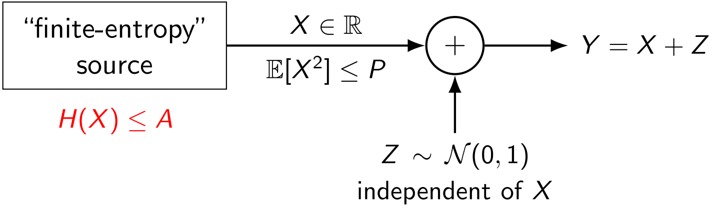
- Optimization 101…thanks to a different problem:



Outline

Input-entropy-constrained channel capacity

- A contraction problem in communication



$$C_H(A,P) = \sup_{\substack{\mathrm{P}_X:\ \mathbb{E}[X^2] \leq P \\ H(X) \leq A}} I(X;Y)$$

Input-entropy-constrained channel capacity

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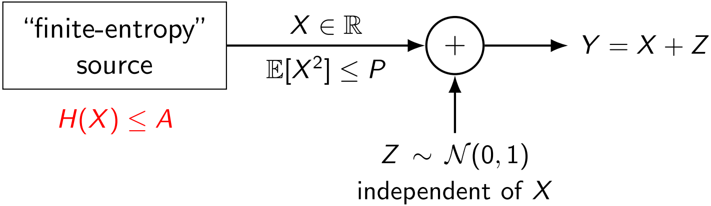

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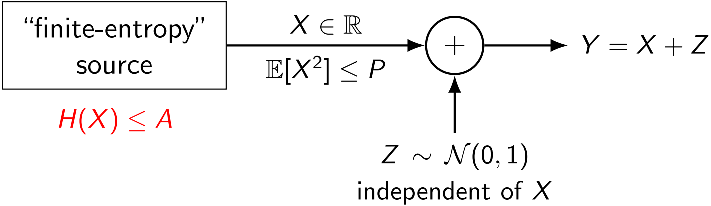
- Cardinality bounds?

- A contraction problem in communication



$$C_H(A,P) = \sup_{\substack{P_X:\ \mathbb{E}[X^2] \leq P \\ H(X) \leq A}} I(X;Y)$$

- Cardinality bounds? Finite support?

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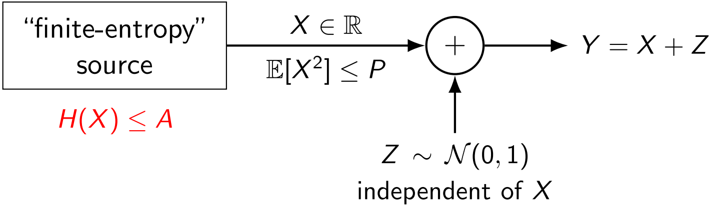

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Aside on data processing inequalities

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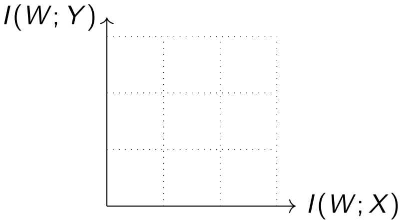

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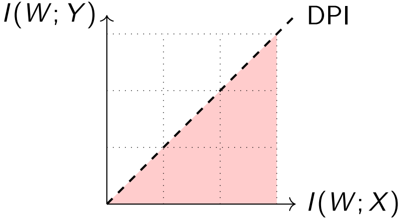

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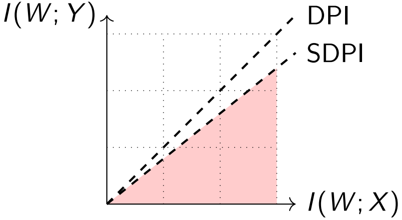

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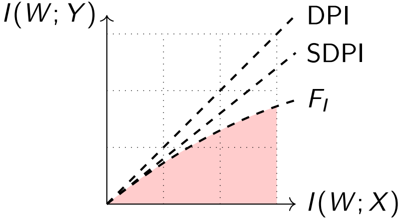

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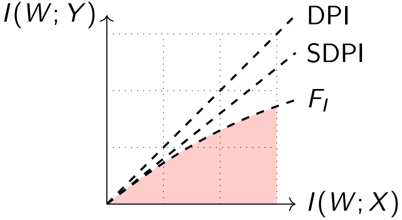

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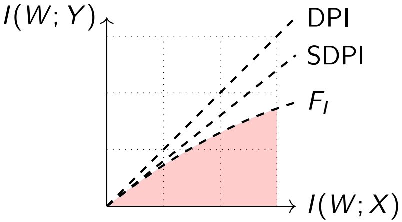

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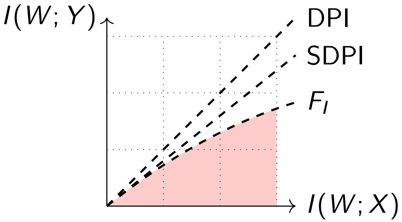

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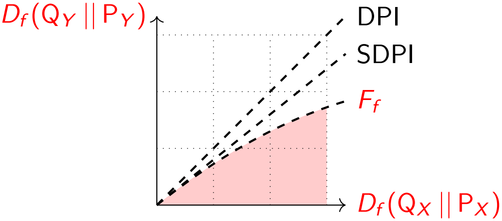

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Outline

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Outline

1. Prompt compression for black-box language models
2. Input-entropy-constrained capacity
3. Joint range of divergences
4. Closing remarks

In closing. . .

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Thank you!