A moment-matching problem with an entropy constraint

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July 2, 2025

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thanks to Shlomo Shamai and Emre Telatar

(Girish–Shamai–Telatar, ISIT 2025)

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- given f_W and constraints on X, how large can k be?

Outline

1. Moment problems
2. Entropy-constrained version
3. Application to a hypothesis testing problem

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  finite support: at most $\lfloor k/2 \rfloor + 1$ atoms (if it exists)

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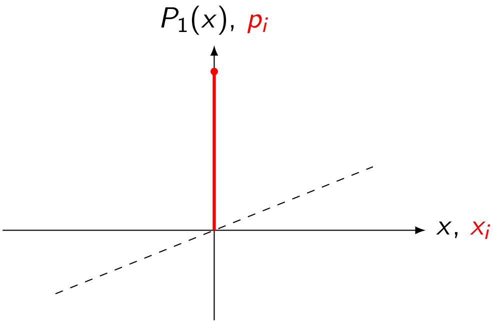

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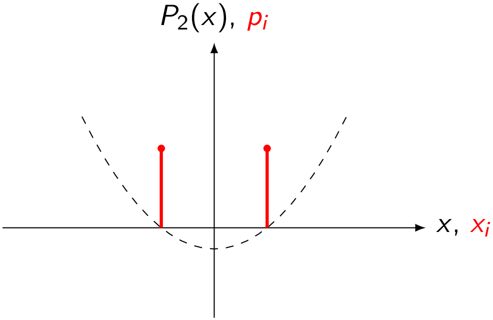

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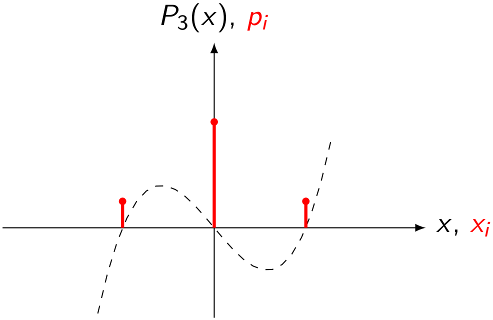

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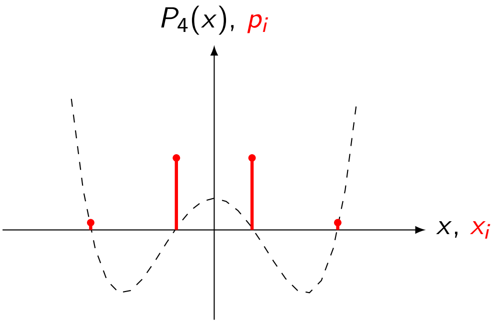

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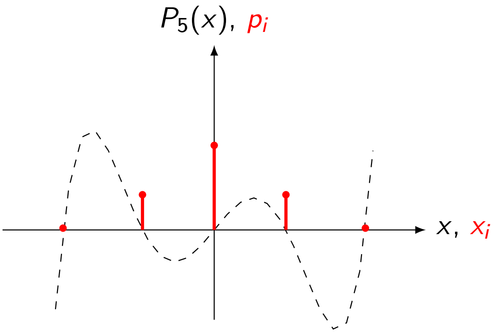

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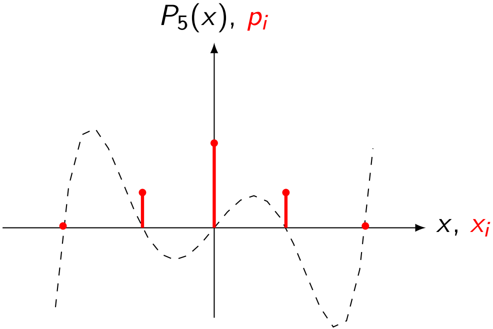

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- optimal trade-off of atoms to moments matched

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- how many moments can we match if entropy is at most h then?

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- check: $s_1, \ldots, s_4$ "valid" iff $\epsilon > \eta_W$, but s_1, s_2, s_3 always "valid" $\square$

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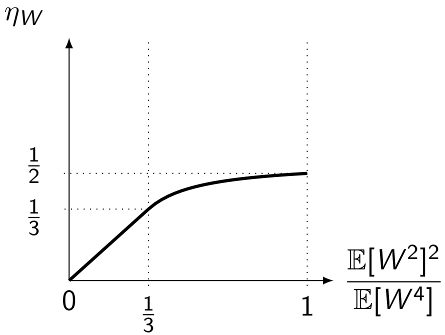

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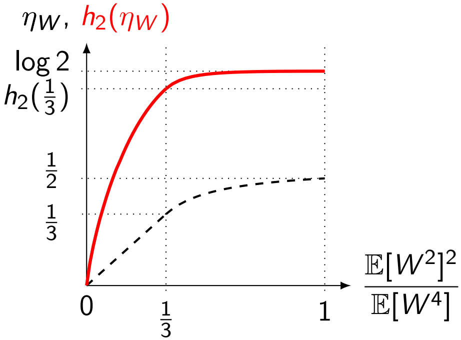

Outline

Testing against Gaussian in large noise

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