Hypercontractivity and Information Theory

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EPFL Information Processing Group

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Outline

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1. From Hölder to hypercontractivity
2. Application to probability theory
3. Connection with information measures

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1. From Hölder to hypercontractivity
2. Application to probability theory
3. Connection with information measures
4. Application to an information-theoretic problem

Outline

Background papers

- Ahlswede-Gács '76
- Nair '14
- Anantharam-Gohari-Kamath-Nair '13a

Background papers

- Ahlswede-Gács '76

Rudolf Ahlswede and Peter Gacs. "Spreading of Sets in Product Spaces and Hypercontraction of the Markov Operator". In: *The Annals of Probability* (1976)

- Nair '14

Chandra Nair. "Equivalent formulations of hypercontractivity using information measures". In: *Proc. International Zurich Seminar on Communications*. 2014

- Anantharam-Gohari-Kamath-Nair '13a

Venkat Anantharam, Amin Aminzadeh Gohari, Sudeep Kamath, and Chandra Nair. "On hypercontractivity and the mutual information between Boolean functions". In: *Proc. Allerton Conference on Communication, Control, and Computing*. 2013

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Hölder's inequality for probability measures

$$\mathbb{E}[|f(X)g(Y)|] \leq \|f(X)\|_{p'} \|g(Y)\|_{p}, \ p \geq 1$$

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Hölder conjugates

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p-norm

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- $\|Z\|_p \triangleq \mathbb{E}[|Z|^p]^{\frac{1}{p}} \longleftarrow$ p-norm of Z

An equivalent formulation

$$\|\mathbb{E}[g(Y) \mid X]\|_p \leq \|g(Y)\|_p, \; p \geq 1$$

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Hypercontractivity parameters

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Indecomposable r.v.s

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There do NOT exist nontrivial sets A and B such that

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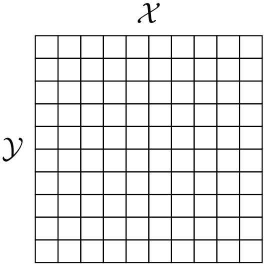

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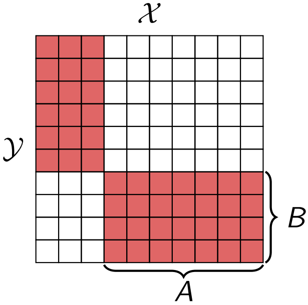

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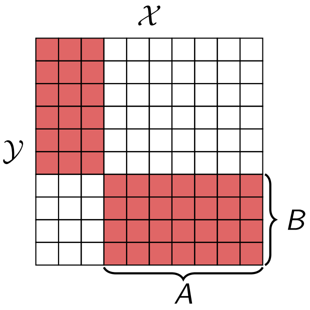

$\longrightarrow$ decomposable

Probability of decoding sets

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$$\mathbb{P}(Y^n \in B_n) \geq \mathbb{P}(Y^n \in B_n \mid X^n \in A_n)^p \, \mathbb{P}(X^n \in A_n)^r .$$

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$$-\frac{1}{n} \log \mathbb{P}(Y^n \in B_n) \leq -\frac{r}{n} \log \mathbb{P}(X^n \in A_n) - \frac{p}{n} \lambda.$$

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$$-\frac{1}{n} \log \mathbb{P}(Y^n \in B_n) \lesssim -\frac{r}{n} \log \mathbb{P}(X^n \in A_n).$$

Outline

Characterizations of r^*...

Theorem (Ahlswede-Gács '76)

$$r^*(X;Y) = \sup_{\nu_X : \substack{\nu_X \neq \mu_X, \\ \nu_X \ll \mu_X}} \frac{D_{\mathrm{KL}}(\nu_Y \,||\, \mu_Y)}{D_{\mathrm{KL}}(\nu_X \,||\, \mu_X)}.$$

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Theorem (Anantharam-Gohari-Kamath-Nair '13b[†])

$$r^*(X;Y) = \sup_{\nu_{UX} : \substack{I(U;X) > 0, \\ U - X - Y}} \frac{I(U;Y)}{I(U;X)}.$$

[†] Venkat Anantharam, Amin Aminzadeh Gohari, Sudeep Kamath, and Chandra Nair. "On Maximal Correlation, Hypercontractivity, and the Data Processing Inequality studied by Erkip and Cover". In: *CoRR* (2013)

...and r_p

Theorem (Nair '14)

$$r_p(X;Y) = \sup_{\nu_{XY}:\substack{\nu_{XY}\neq\mu_{XY},\\ \nu_{XY}\ll\mu_{XY}}} \frac{D_{\mathrm{KL}}(\nu_Y \,||\, \mu_Y)}{D_{\mathrm{KL}}(\nu_X \,||\, \mu_X) + p\begin{pmatrix} D_{\mathrm{KL}}(\nu_{XY} \,||\, \mu_{XY}) \\ - D_{\mathrm{KL}}(\nu_X \,||\, \mu_X)\end{pmatrix}}.$$

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A conjecture on Boolean functions

Conjecture (Courtade-Kumar '14†)

† Thomas A. Courtade and Gowtham R. Kumar. "Which Boolean Functions Maximize Mutual Information on Noisy Inputs?" In: *IEEE Transactions on Information Theory* (2014)

A conjecture on Boolean functions

$X_i, Y_i \sim \text{Bern}\left(\frac{1}{2}\right)$,
$\mathbb{P}(X_i \neq Y_i) = \alpha$.

Conjecture (Courtade-Kumar '14†)

For $\{(X_i, Y_i)\}_{i=1}^{n}$ *i.i.d.* DSBS(α)

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A conjecture on Boolean functions

Conjecture (Courtade-Kumar '14[†])

For $\{(X_i, Y_i)\}_{i=1}^n$ *i.i.d.* DSBS(α), *for any Boolean functions* b_1, b_2,

$$I(b_1(X^n); b_2(Y^n)) \leq 1 - h_2(\alpha).$$

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A stronger conjecture on r^* of binary r.v.s

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- $W - X^n - Y^n - Z \implies r^*(W;Z) \leq r^*(X^n;Y^n)$;
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- $(X_i, Y_i) \sim \mathrm{DSBS}(\alpha) \implies r^*(X_i;Y_i) = (1-2\alpha)^2$.

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Takeaways

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- Connections to information-theoretic quantities
- Applications interesting, but...
- ...more work to be done!

Research proposal

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- Computable characterizations of r_p and r^*

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- Closed form expressions in particular cases

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- Computable characterizations of r_p and r^*
- Closed form expressions in particular cases
- More connections and applications to information theory

Thank you!