Operational Interpretation of Rényi Information Measures via Composite Hypothesis Testing

Adway Girish
Information Theory Laboratory

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Outline

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Simple hypothesis testing

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X^n

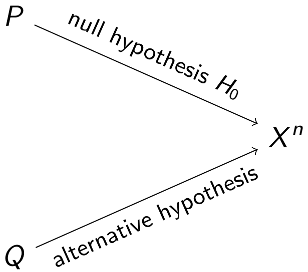
Simple hypothesis testing

P

X^n

Q

Simple hypothesis testing



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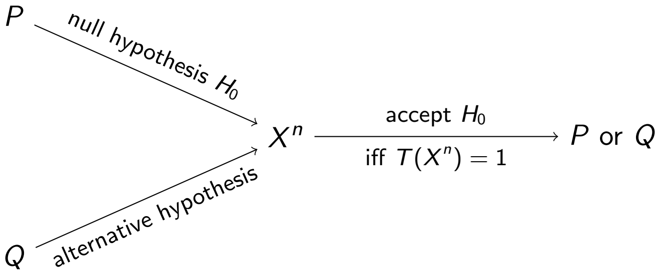

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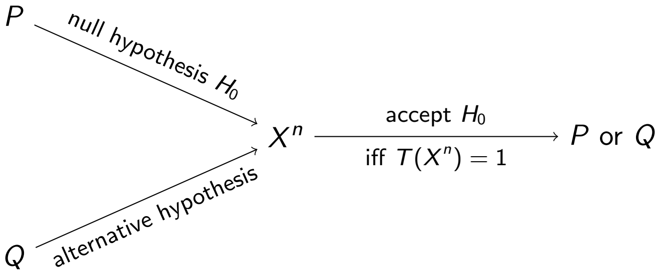

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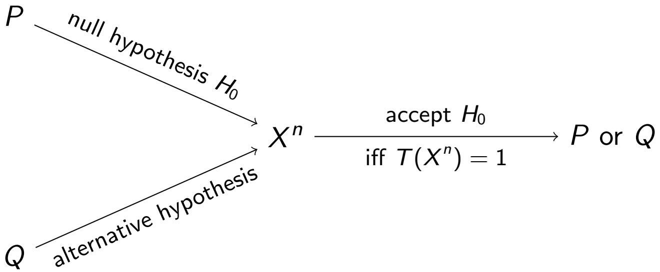

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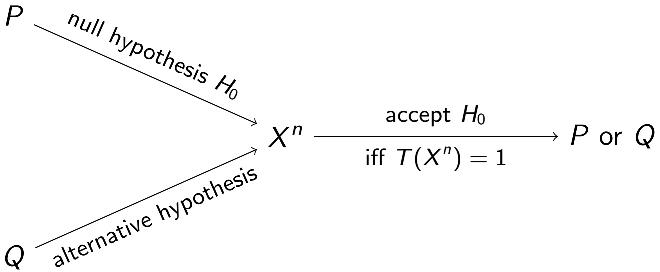

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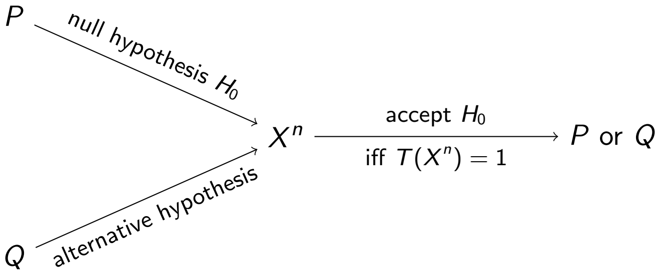

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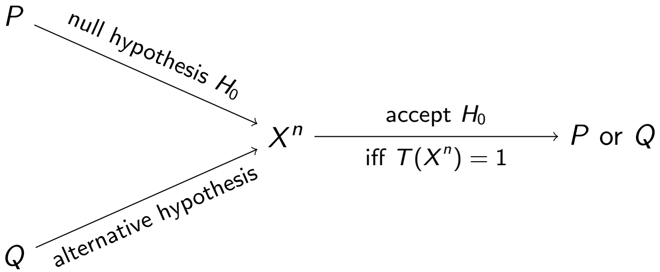

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- If $\mathrm{p}_n \le \epsilon$, then the optimal $\mathrm{q}_n = \exp\big(-nI(X;Y) + o(n)\big)$.
- Operational interpretation of $I(X;Y)$.

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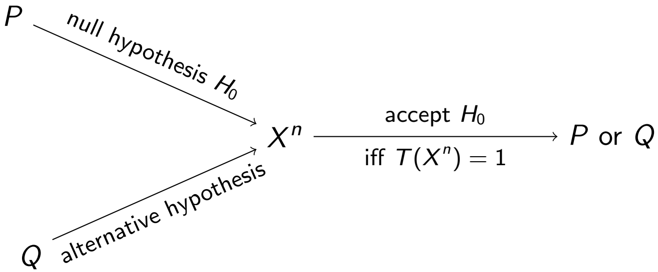
Sibson's mutual information[†]

[†] Robin Sibson. "Information radius". In: *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 2 (1969)

Simple hypothesis testing again



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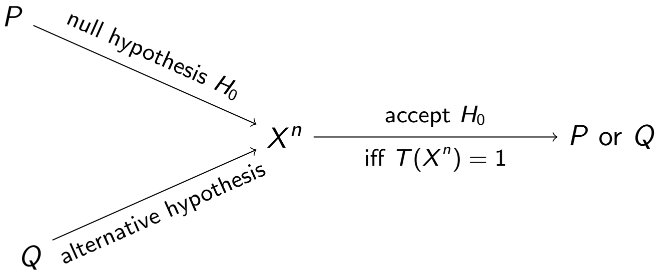

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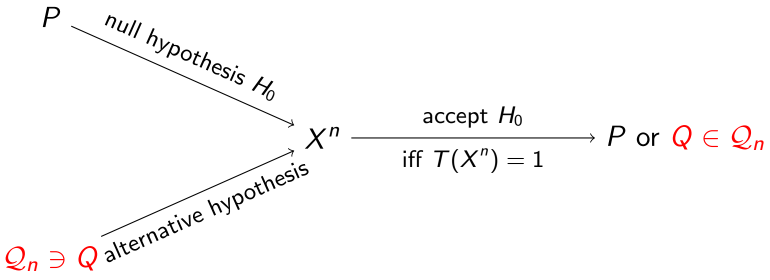

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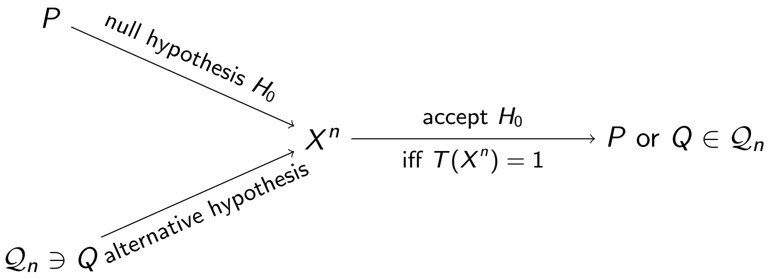

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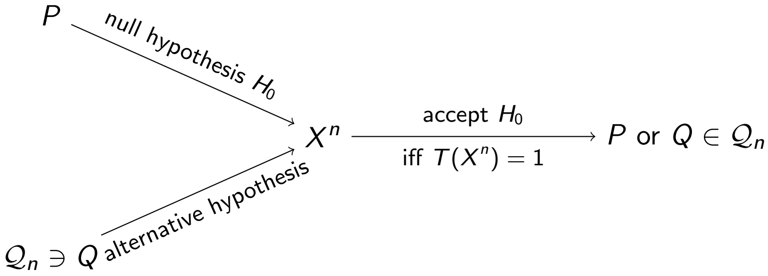

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  but *only* under some conditions on $\mathcal{Q}_n$.

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